

A holographic correspondence from tensor network states

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We introduce a toy holographic correspondence based on the multi-scale entanglement renormalization ansatz (MERA) representation of ground states of local Hamiltonians. Given a MERA representation of the ground state of a local Hamiltonian acting on an one dimensional ‘boundary’ lattice, we lift it to a tensor network representation of a quantum state of a dual two dimensional ‘bulk’ hyperbolic lattice. The dual bulk degrees of freedom are associated with the bonds of the MERA, which describe the renormalization group flow of the ground state, and the bulk tensor network is obtained by inserting tensors with open indices on the bonds of the MERA. We explore properties of *copy bulk states*—particular bulk states that correspond to inserting the *copy tensor* on the bonds of the MERA. We show that entanglement in copy bulk states is organized according to holographic screens, and that expectation values of certain extended operators in a copy bulk state, dual to a critical ground state, are proportional to n -point correlators of the critical ground state. We also present numerical results to illustrate e.g. that copy bulk states, dual to ground states of several critical spin chains, have exponentially decaying correlations, and that the correlation length generally decreases with increase in central charge for these models. Our toy model illustrates a possible approach for deducing an emergent bulk description from the MERA, in light of the on-going dialogue between tensor networks and holography.

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I. INTRODUCTION

In recent years, renormalization group ideas have been playing an important role in the efficient classical simulation of quantum many-body systems at low energies. By properly identifying and renormalizing the high energy degrees of freedom one can build an efficient representation of the ground state of the system. A prominent example is *entanglement renormalization*—a real space renormalization group (RG) transformation on a lattice, which removes local entanglement before coarse-graining the system [1]. Entanglement renormalization forms the basis of the *multi-scale entanglement renormalization ansatz* (MERA): an efficient tensor network representation of ground states of local Hamiltonians on a lattice [2].

The MERA representation of the ground state of a local Hamiltonian, acting on an one dimensional (1D) quantum lattice, consists of a two dimensional (2D) hyperbolic tensor network. The 2D tensor network also encodes the RG flow of the ground state, and the extra dimension in the tensor network description corresponds to length scale in the 1D system. The MERA has been successfully applied to simulate ground states of several quantum lattice systems, and seems particularly well suited for critical systems, which are described by conformal field theories (CFTs) in the continuum [3].

On the other hand, the AdS/CFT correspondence suggests an intimate relationship between RG and emergent gravity in anti-deSitter (AdS) spacetimes. The AdS/CFT correspondence [4, 5]—a concrete realization of the holographic principle—is an equivalence between certain quantum gravity theories in $d + 1$ -dimensional AdS spacetimes and CFTs that live on the d -dimensional boundary of the AdS spacetime. The correspondence es-

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entially translates an RG description of the CFT to a bulk gravity theory in AdS geometry. In particular, the extra dimension of the AdS geometry is identified with length scale in the CFT.

Recently, it has been conjectured [6–9], first in Ref. 6, that the MERA realizes certain features of the AdS/CFT correspondence. In particular, it has been proposed that the MERA tensor network can be viewed as a discretization either of a spatial slice of an emergent AdS spacetime [6] or of the abstract space of geodesics of a spatial slice of AdS [8]. However, while it is understood how the MERA encodes an 1D critical ground state, there is no conclusive view on how it could also potentially encode a dual description of an emergent 2D system. An even more basic problem is to identify the relevant bulk degrees of freedom.

In this paper, we construct a toy holographic correspondence from the MERA representation of ground states of 1D local Hamiltonians to illustrate a possible way in which the MERA could encode a dual description. Specifically, we describe how to ‘lift’ a MERA representation of an 1D ground state to a quantum state of a 2D quantum lattice. The two states are seen to live on the boundary and in the bulk of a 2D manifold respectively. We refer to this quantum state correspondence, mediated by a tensor network, as holographic because it is guided by and implements certain basic features of the AdS/CFT correspondence. Namely, (i) the dual bulk degrees of freedom describe the RG flow of the 1D system, (ii) the 1D ground state is dual to a 2D *quantum state* in the bulk [10], which may reduce to a classical description in some limit, (iii) the dual bulk state is completely determined from boundary state (since the promise of holography is that the physical states of gravity simply correspond to the states of dual CFT), and (iv) if the ground state has a global internal symmetry described by a group \mathcal{G} , the dual bulk state has a local gauge symmetry \mathcal{G} . In this paper, we introduce an ansatz for the dual bulk state that exhibits the features (i) – (iii) listed above, while we extend the formalism to incorporate feature (iv) in a following paper [11].

We make a few remarks pertaining to the feature (ii). The local Hamiltonians that we consider here may be critical and described by CFTs with a *small* central charge. (In fact, in practice the MERA has been mostly applied to simulate ground states of CFTs that have a small central charge.) In the AdS/CFT correspondence, a small boundary central charge (e.g. of order 1) generally corresponds to *quantum* gravity in the bulk [12]. For example, Ref. 13 presents a holographic description of the 1D quantum critical Ising model, which has central charge equal to $\frac{1}{2}$. Indeed, there the authors match the partition function of the Ising model to a dual quantum gravity partition function, obtained by summing over all bulk geometries (gravitational fields) that are compatible with the asymptotic constraints imposed by the boundary theory. This motivates us to derive a dual *quantum* state from the MERA, and subsequently we focus on

the entanglement and correlation properties of the dual bulk state. In particular, here we do not attempt to deduce a semi-classical bulk geometry from the MERA (or even from the dual bulk states), or assume any particular spacetime interpretation of the MERA’s hyperbolic geometry.

The bulk ansatz is described by a tensor network that is obtained by inserting tensors with open indices on the bonds of the MERA. We explore a particular subset of bulk states—*copy bulk states*—which have interesting properties. For example, entanglement in a copy bulk state is organized according to ‘holographic screens’. A holographic screen consists of sites of the bulk lattice that are located along a 1D path anchored at two boundary locations, such that the entanglement entropy of the sites on the screen is *equal* to the entanglement entropy of the 2D bulk region enclosed between the screen and the boundary of the geometry. We also consider copy bulk states dual to an 1D critical ground state, and show that the bulk expectation values of certain *extended* operators are proportional to n -point correlators of the (boundary) critical ground state. This caricatures the prescription to calculate boundary correlators in AdS/CFT by evaluating *Witten diagrams* [5, 14].

The paper is organized as follows. In Sec. II, we briefly review the MERA representation of ground states of infinite 1D quantum lattice systems. In Sec. III we introduce an ansatz for the bulk state dual to a given 1D ground state and also copy bulk states—particular states belonging to the ansatz. In Sec. IV we describe how copy bulk states exhibit holographic screens. In Sec. V we describe a simple prescription to obtain n -point correlators of scaling operators and the block Reyni entanglement entropy of an 1D critical ground state from a dual copy bulk state. In Sec. VI, we present numerical results pertaining to the bulk entanglement and correlations in copy bulk states, dual to ground states of several exactly solvable critical spin chains. We conclude with a summary and outlook in Sec. VII. The appendices contain proofs and other technical details that are not covered in the main text for simplicity of presentation.

II. BOUNDARY STATE

Consider an infinite 1D lattice \mathcal{L} , each site of which is described by a χ -dimensional Hilbert space \mathbb{V} . Lattice \mathcal{L} is equipped with the action of a local, translational invariant Hamiltonian \hat{H} , which may be gapped or critical (gapless). We are interested in the ground state $|\Psi^{\text{bound}}\rangle$ of \hat{H} . The superscript ‘bound’ appears in anticipation that the ground state will play the role of the boundary state in our holographic correspondence. In this paper, we represent $|\Psi^{\text{bound}}\rangle$ by means of an infinite MERA tensor network, depicted in Fig. 1. The MERA representation of the ground state of a given local Hamiltonian can be obtained by means of e.g. the variational energy minimization algorithm [15].

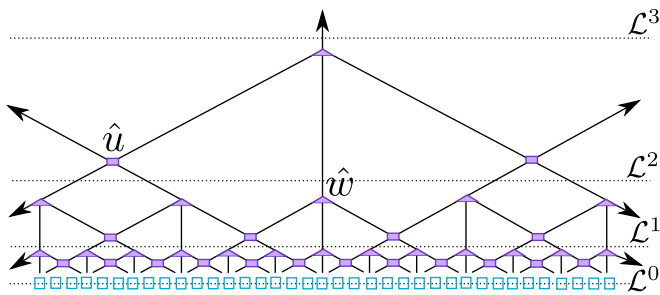


FIG. 1. (Color online) Graphical representation of the MERA tensor network representation of a quantum many-body state of an infinite lattice \mathcal{L} ($\cong \mathcal{L}_0$). Arrows in the figure indicate that the tensor network extends infinitely in the upward vertical and both left, right horizontal directions. Each open index of the tensor network labels an orthonormal basis on a different site (blue squares) of \mathcal{L} . The dotted horizontal lines separate the tensor network into layers of tensors, and coincide with a sequence of increasing coarse-grained lattices: $\mathcal{L}^0 \rightarrow \mathcal{L}^1 \rightarrow \mathcal{L}^2 \dots$. The vertical direction of the tensor network corresponds to length scale, namely, after discarding the bottom layers the residual tensor network describes the many-body state at a coarser length scale.

The *open indices* of the tensor network are associated with the sites of \mathcal{L} , namely, each open index labels an orthonormal basis on a different site of \mathcal{L} . On the other hand, the *bond indices*—indices that connect the tensors in the network—carry the entanglement and correlations in the quantum state. The probability amplitudes for the state $|\Psi^{\text{bound}}\rangle$, in the basis labelled by the open indices, are formally obtained by contracting all the tensors of the infinite tensor network, which involves summing the bond indices.

The MERA representation also describes the RG flow of the ground state. Each layer of MERA tensors, separated by dotted lines in Fig. 1, implements a real space RG transformation—known as *entanglement renormalization*—that maps a lattice \mathcal{L}^k with L ($\rightarrow \infty$) sites to a coarse-grained lattice \mathcal{L}^{k+1} with $L/3$ sites. The MERA tensors are chosen so that the RG transformation preserves the ground subspace at each step. Subsequent RG steps generate a sequence of increasingly coarse-grained lattices: $\mathcal{L}^0 \rightarrow \mathcal{L}^1 \rightarrow \mathcal{L}^2 \dots$ where $\mathcal{L}^0 \cong \mathcal{L}$ is the ultraviolet lattice. Therefore, the extra dimension of the tensor network corresponds to length scale. In particular, the residual tensor network obtained after discarding one or more bottom layers of the MERA furnishes a representation of the ground state on a coarse-grained lattice.

If, in addition to translation invariance, the ground state is also *scale-invariant*—namely, it remains invariant under the RG (entanglement renormalization) transformations—then its MERA representation is composed of copies of the same two tensors \hat{u} and \hat{w} throughout the tensor network. This leads to a very compact description of the infinite ground state, namely, the entire state is completely specified by the two tensors \hat{u} and \hat{w} . In the rest of the paper, we will assume that

the Hamiltonian \hat{H} (and its ground state $|\Psi^{\text{bound}}\rangle$) is also scale-invariant, and that the MERA representation of the ground state is composed of copies of tensors \hat{u} and \hat{w} .

The MERA tensors \hat{u} and \hat{w} are constrained to be isometries satisfying:

$$\sum_{kl} (\hat{u})_{kl}^{ij} (\hat{u}^\dagger)_{i'j'}^{kl} = \delta_{i'}^i \delta_{j'}^j, \quad \sum_{jkl} (\hat{w})_{jkl}^i (\hat{w}^\dagger)_{i'}^{jkl} = \delta_{i'}^i, \quad (1)$$

where $i, j, k, l, i', j' \in \{1, 2, \dots, \chi\}$. Consequently, the reduced density matrix of any site on the lattice does not depend on all the MERA tensors, but only on a subset of them; this subset of tensors is called the *causal cone* of the site. The number of tensors in the causal cone that are counted at any given length scale is bounded (less than or equal to 3). Furthermore, the reduced density matrix of multiple sites depends only on tensors belonging to the union of the respective one-site causal cones, which merge at a sufficiently large length scale. Thanks to these properties, expectation values can be efficiently computed from an infinite scale-invariant MERA tensor network [2, 15].

III. AN ANSATZ FOR THE DUAL BULK STATE

In this section, we construct a 2D ‘bulk’ description dual to the 1D ground state $|\Psi^{\text{bound}}\rangle$. Our construction is motivated by the basic features of the AdS/CFT correspondence listed (i)-(iv) in the introduction.

To begin with, what are the dual bulk degrees of freedom? In a tensor network representation of a quantum many-body state, the bond indices of the tensor network are treated differently from the open indices. While the open indices of the tensor network are associated with the degrees of freedom of the state, the bond indices carry the entanglement and correlations in the state. Here we propose to construct a dual emergent description of a tensor network state by associating the bond indices with emergent degrees of freedom, and thus, figuratively speaking, treating the bond indices on an equal footing as the open indices. The bond indices of the MERA, in particular, are also associated with the renormalized sites. In the AdS/CFT correspondence, the RG flow of the CFT plays an instrumental role in the dual bulk description. This further motivates associating the dual bulk degrees of freedom with the bonds of the MERA.

Let us embed the MERA tensor network, which represents $|\Psi^{\text{bound}}\rangle$, in a 2D manifold with a boundary, such that the open indices and bond indices of the tensor network appear at the boundary and in the bulk of the manifold respectively. Construct a 2D quantum lattice \mathcal{M} on the manifold by locating a site—described by the χ -dimensional Hilbert space \mathbb{V} —on every bond of the (embedded) tensor network, see Fig. 2(a). Lattice \mathcal{M} is simply a collation of the degrees of freedom that describe the RG flow of the boundary state, and also inherits the hyperbolic geometry of the tensor network [16].

Next, let us insert a three index tensor $(\hat{c})_{i'o}^i$, where $i, i', o \in \{1, 2, \dots, \chi\}$, on each bond of the MERA as shown in Fig. 2(b), and use the open index o to label an orthonormal basis on the site of \mathcal{M} located on the bond. The new tensor network, which is composed of the ground state tensors \hat{u} and \hat{w} and copies of the bond tensor \hat{c} , encodes a quantum state $|\Psi^{\text{bulk}}\rangle$ of \mathcal{M} . Analogous to the MERA, the probability amplitudes of $|\Psi^{\text{bulk}}\rangle$ are obtained by contracting together all the tensors of the new tensor network. Thus, we have ‘lifted’ the MERA description of an 1D ground state $|\Psi^{\text{bound}}\rangle$ to a 2D quantum state $|\Psi^{\text{bulk}}\rangle$ belonging to the lattice \mathcal{M} . We refer to the new tensor network as the *lifted MERA*. Since we have embedded the MERA in a 2D manifold with a boundary, the two states $|\Psi^{\text{bound}}\rangle$ and $|\Psi^{\text{bulk}}\rangle$ are seen to live at the boundary and in the bulk of the hyperbolic lattice \mathcal{M} . (Here we identify the sites located at the boundary of \mathcal{M} with the sites of lattice \mathcal{L} . For example, the site of \mathcal{M} located at $(x, 0)$ is identified with the site of \mathcal{L} located at x . Note that the bulk state $|\Psi^{\text{bulk}}\rangle$ also has support on the boundary sites of \mathcal{M} .)

States $|\Psi^{\text{bound}}\rangle$ and $|\Psi^{\text{bulk}}\rangle$ constitute our holographic correspondence, mediated by the MERA tensor network composed of tensors \hat{u} and \hat{w} . For a more general construction, one may also pre-process the MERA by contracting and/or decomposing some of its tensors before lifting it to a 2D quantum state as described here, see Appendix C. Here, we have inserted copies of the same tensor \hat{c} on all the bonds of the MERA to build a locally uniform bulk tensor network, which ensures that the corresponding bulk state $|\Psi^{\text{bulk}}\rangle$ is invariant under translations along some directions.

Since we have still not fixed the bond tensor \hat{c} , the lifted MERA describes a set of states, one for each choice of the bond tensor, on the lattice \mathcal{M} . The lifted MERA is our *ansatz* for the holographic dual of the 1D ground state $|\Psi^{\text{bound}}\rangle$. Requiring that the dual bulk state must completely derive from the ground state, as in the AdS/CFT correspondence, we impose that tensor \hat{c} must either be a constant or a function only of the ground state tensors \hat{u} and \hat{w} . Additional guidelines from AdS/CFT may be used to further fix the properties of \hat{c} . For example, if the boundary state $|\Psi^{\text{bound}}\rangle$ has a global symmetry \mathcal{G} , we require \hat{c} to transform under the action of the symmetry in a particular way, in order to ensure that the corresponding bulk state has a local symmetry \mathcal{G} [11] (thus implementing feature (iv) of the AdS/CFT correspondence listed in the introduction).

The bulk states described by a lifted MERA contain only a limited entanglement. Given a subsystem of the bulk lattice \mathcal{M} , we define its perimeter and area as the number of sites that lie at the subsystem’s boundary and inside the subsystem respectively. For bulk states described by a lifted MERA, the entanglement entropy of a sufficiently large subsystem scales at most as the perimeter of the subsystem, as proved in Appendix B. Such an entanglement scaling is ubiquitous in condensed matter physics where it is called ‘(boundary) area law entan-

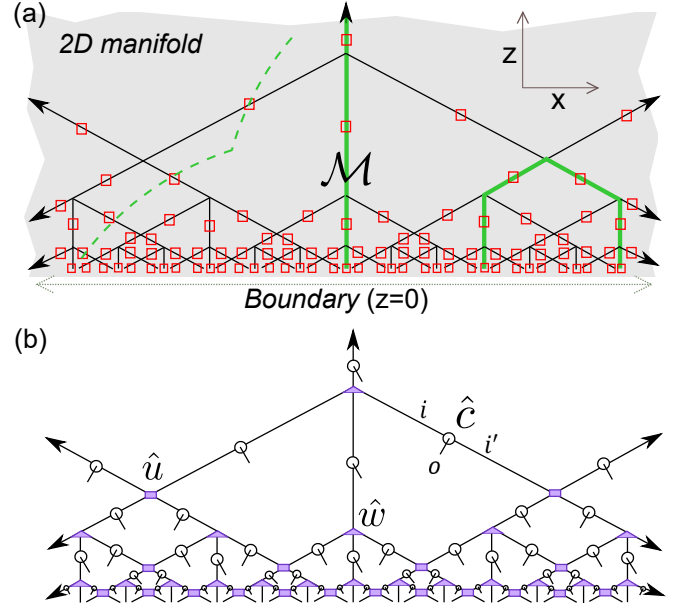


FIG. 2. (Color online) (a) The dual 2D bulk lattice \mathcal{M} , constructed by embedding the MERA in a 2D manifold with a boundary (at $z = 0$), and locating a site (red square) of \mathcal{M} on every bond of the MERA. The green solid paths are *graph geodesics*, namely, geodesic paths along the edges of the graph underlying the MERA. Also shown is a path (green dashes) in the ambient manifold that intersects only the edges of the MERA graph. (b) The lifted MERA, our ansatz for the holographic dual state, obtained by inserting a 3-index tensor $(\hat{c})_{i'o}^i$ on every bond of the MERA. By using each open index of the tensor network (e.g. index o) to label an orthonormal basis on a different site of \mathcal{M} , the lifted MERA encodes a quantum state of \mathcal{M} .

glement scaling’ and is commonly exhibited by ground states of local, low-dimensional quantum lattice systems. In contrast, one expects the subsystem entanglement entropy of generic states of \mathcal{M} to scale as the subsystem’s area.

A. Copy bulk states

In the remainder of the paper, we explore properties of particular bulk states belonging to the ansatz described above. Specifically, we consider bulk states obtained by choosing \hat{c} simply as the *copy tensor*:

$$(\hat{c})_{oi'}^i = \begin{cases} 1, & \text{if } i = i' = o, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

(This corresponds to fixing a basis on the bulk sites as discussed later in this section.) Colloquially, \hat{c} copies a bond index of the MERA to an open index of the lifted MERA, and is perhaps the simplest choice to lift a tensor network representation. The boundary state $|\Psi^{\text{bound}}\rangle$ can be recovered from a dual copy bulk state $|\Psi^{\text{bulk}}\rangle$, simply

as

$$|\Psi^{\text{bound}}\rangle = \left(\bigotimes_k \hat{P}_k\right)|\Psi^{\text{bulk}}\rangle \quad \hat{P}_k = \sum_{j=1}^{\chi} |j\rangle\langle j|, \quad (3)$$

where \hat{P}_k projects the bulk site located on bond k to the state $|+\rangle \equiv \sum_{j=1}^{\chi} |j\rangle$. Analogous to the MERA, a copy-lifted MERA is also endowed with a causal cone structure that allows for efficient computation of expectation values in the bulk, as described in Appendix A.

Given a MERA representation of a quantum many-body state, one can obtain another equivalent MERA representation of the state by inserting a resolution of identity $\hat{M}_k \hat{M}_k^{-1}$ on bond k , and multiplying the matrices \hat{M}_k and \hat{M}_k^{-1} respectively with the two tensors that are connected by the bond. This *bond freedom* is an intrinsic property of tensor network representations of quantum many-body states. The two MERAs are an equivalent representation of the state since the expectation value of any observable is equal in both representations. (Obtaining an expectation value from a tensor network state involves contracting all the bond indices of the tensor network, and \hat{M}_k is multiplied with \hat{M}_k^{-1} in the process.)

Clearly, inserting the copy tensor defined in Eq. (2) selects out a particular MERA representation of the ground state—the one whose tensors are expressed in the basis in which the copy tensor has the components of Eq. (2). Consider copy bulk states $|\Psi\rangle$ and $|\Psi'\rangle$ that are obtained by lifting MERA tensor networks \mathcal{T} and \mathcal{T}' respectively. Here MERA \mathcal{T}' is obtained by transforming the tensors of \mathcal{T} by means of *non-diagonal* unitary bond transformations (\hat{M}_k 's) as described above. In this case, even though the two MERAs \mathcal{T} and \mathcal{T}' describe the same quantum many-body state, the two corresponding copy bulk states $|\Psi\rangle$ and $|\Psi'\rangle$ are generally different e.g. they have different entanglement. This is because the copy tensor \hat{c} ‘commutes’ with only *diagonal* matrices, namely, a contraction of \hat{c} with a diagonal matrix on any index is equal to a contraction of \hat{c} with the same diagonal matrix on a different index. This implies that $|\Psi\rangle$ and $|\Psi'\rangle$ are not related to each other by one-site unitary rotations on the bulk lattice, and therefore they have different entanglement. Thus, our bulk construction generally lifts a given ground state to a set of different copy bulk states.

In such a scenario, how can one compare bulk states corresponding to different boundary states? For example, one could be interested in probing if the bulk entanglement depends on the boundary central charge, see Sec. VI. A possible approach to compare bulk states corresponding to different boundary states is to average the bulk properties over all the different copy bulk states that are dual to the ground state. Another possibility is to compare the *statistics* of the bulk properties by randomly sampling from the different copy bulk states. (See Ref. 11 for some statistics of the bulk properties). On the other hand, one may take the view that a given ground state must lift to only one dual bulk state to begin with. A possible way to achieve this, within the framework in-

troduced thus far, is to constrain the intrinsic bond freedom in MERA representations by demanding that the tensors fulfill additional constraints.

For example, in Ref. 11 we describe how the formalism presented in this paper can be extended to implement the holographic translation of a boundary on-site global symmetry to a bulk local gauge symmetry. This is achieved by partially constraining the bond freedom, which illustrates that constraining the bond freedom may indeed be useful (or even necessary) to implement certain features of the AdS/CFT in our holographic correspondence. In Appendix C we introduce two modified MERA representations that have a significantly constrained bond freedom, as compared to the standard MERA representation which is reviewed in Sec. II. One of these constrained MERA representations was used to obtain the numerical results presented in Sec. VI. However, these constrained MERA representations are not directly motivated from AdS/CFT. Nonetheless, they illustrate a possible generalization of our bulk construction to control the number of different copy bulk states dual to a given ground state.

It is also possible that the different copy bulk states, dual to a ground state, obtained here are related to one another by an unidentified (emergent) bulk symmetry, and are thus equivalent holographic duals of the ground state. Or that the relevant dual bulk state is a certain superposition of the different copy bulk states that is determined by some holography inspired bulk conditions. We leave further exploration of these issues for future work.

IV. HOLOGRAPHIC SCREENS

Let us parameterize the sites of the bulk lattice \mathcal{M} by coordinates (x, z) where, in the boundary description, x labels spatial translations and z corresponds to the length scale (the boundary is located at $z = 0$). Consider two points P_1 and P_2 at the boundary of the ambient manifold, in which the (lifted) MERA is embedded. P_1 is located on the line segment between bulk sites $(x-1, 0)$ and $(x, 0)$, and P_2 is located on the line segment between bulk sites $(x', 0)$ and $(x'+1, 0)$, for some $x \neq x'$. Consider a path \mathcal{P} between points P_1 and P_2 that intersects only the copy tensors of the lifted MERA, as illustrated (green dashes) in Fig. 3. Path \mathcal{P} divides the bulk lattice \mathcal{M} into three parts:

1. an ‘interior’ composed of bulk sites enclosed between the path and the boundary, and including sites located at $(x, 0), (x+1, 0), \dots, (x', 0)$,
2. bulk sites associated with the copy tensors that are intersected by \mathcal{P} , and
3. an ‘exterior’ composed of all remaining bulk sites.

Let us decorate the indices of tensors \hat{u} and \hat{w} with arrows as depicted (red) in Fig. 3(a). If the arrows on all the bond indices located in the immediate exterior of the

path are incoming to the interior then \mathcal{P} can be viewed as a *holographic screen*. It can be shown that \mathcal{P} satisfies (see Appendix D)

$$\hat{\rho}^{\text{screen}} = \hat{R}^\dagger(\hat{\rho}^{\text{interior}})\hat{R}, \quad (4)$$

where $\hat{\rho}^{\text{screen}}$ is the reduced density matrix of the bulk sites intersected by the screen, $\hat{\rho}^{\text{interior}}$ is the reduced density matrix of the bulk sites located in the interior, and \hat{R} is an isometry, namely, $\hat{R}\hat{R}^\dagger = \hat{I}$. Equation 4 implies that the expectation value of an observable $\hat{o}^{\text{interior}}$ acting in the interior is equal to the expectation value of the observable $\hat{o}^{\text{screen}} = \hat{R}^\dagger(\hat{o}^{\text{interior}})\hat{R}$ acting on the screen, since

$$\begin{aligned} \text{Tr}(\hat{\rho}^{\text{screen}}\hat{o}^{\text{screen}}) &= \text{Tr}(\hat{R}^\dagger\hat{\rho}^{\text{interior}}\hat{R}\hat{R}^\dagger\hat{o}^{\text{interior}}\hat{R}), \\ &= \text{Tr}(\hat{\rho}^{\text{interior}}\hat{o}^{\text{interior}}). \end{aligned} \quad (5)$$

Here we used Eq. (4), the fact that $\hat{R}\hat{R}^\dagger = \hat{I}$, and the cyclic property of trace: $\text{Tr}(AB) = \text{Tr}(BA)$.

Thus, the expectation value of any observable supported in the 2D interior region can be calculated from the 1D screen, which encloses the interior. Furthermore, the expectation value of a *local* interior observable equates to the expectation value of a *local* screen observable. Namely, an observable supported on a small number of interior sites maps to an observable that is also supported on a small number of screen sites. This is because \hat{R} is a composition of isometries, each of which act on a small number of sites. (In contrast, the expectation value of a local screen observable $\hat{\omega}^{\text{screen}}$ generally equates to the expectation value of an interior observable $\hat{\omega}^{\text{interior}} = (\hat{R}^\dagger\hat{R})\hat{\omega}^{\text{screen}}(\hat{R}^\dagger\hat{R})$ that is smeared over all the interior degrees of freedom.) From Eq. (4) it also follows that $\hat{\rho}^{\text{screen}}$ and $\hat{\rho}^{\text{interior}}$ have the same eigenvalues, which in turn implies e.g. that the entanglement entropy of all the interior sites *is equal* to the entanglement entropy of all the screen sites, namely,

$$-\text{Tr}(\hat{\rho}^{\text{interior}}\log_2 \hat{\rho}^{\text{interior}}) = -\text{Tr}(\hat{\rho}^{\text{screen}}\log_2 \hat{\rho}^{\text{screen}}). \quad (6)$$

This entanglement feature is compatible, but extends beyond, the entanglement scaling proved in Appendix B.

Let us define the length of path \mathcal{P} as the number of copy tensors that it intersects. It can be easily verified that if \mathcal{P} is a geodesic between the points P_1 and P_2 , namely, \mathcal{P} intersects the smallest possible number of copy tensors then it is necessarily a holographic screen [Fig. 3(a)]. This is because a geodesic path always fulfills the arrow criterion stated previously and therefore satisfies Eq. (4). On the other hand, Fig. 3(b) illustrates a non-geodesic holographic screen.

The presence of holographic screens described here is a generic property of copy bulk states, and seems to imitate the holographic screens—a feature of (quantum) spacetime—that often appear in quantum gravity. We also remark that the MERA tensors located in the interior of a holographic screen considered here compose a *local conformal transformation* on the boundary state

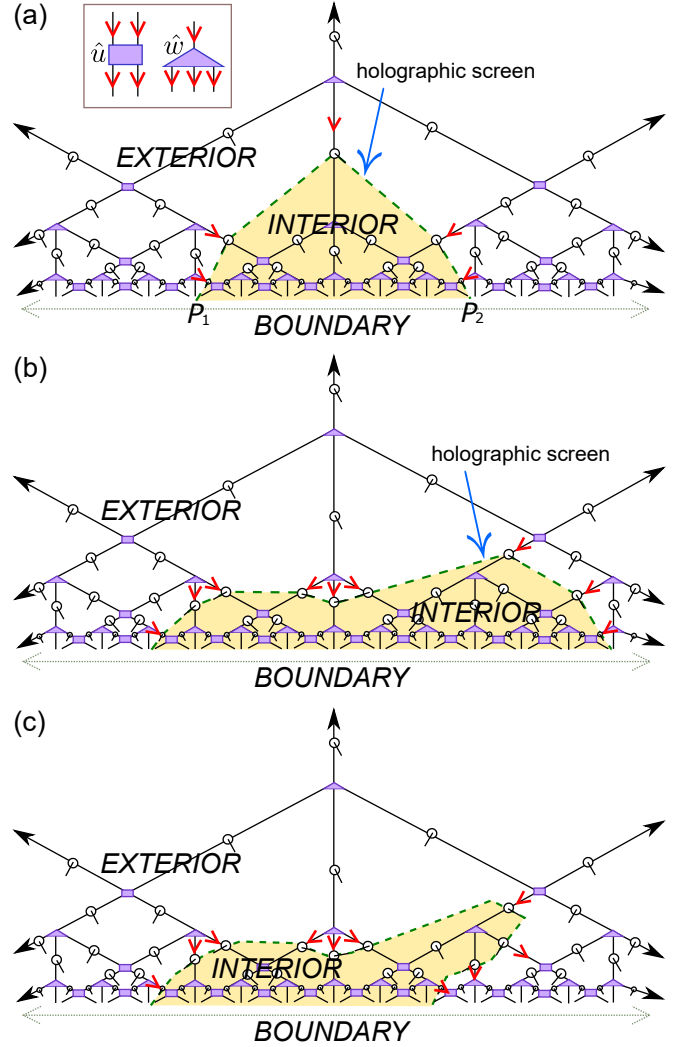


FIG. 3. (Color online) (a,b) Examples of holographic screens in a copy-lifted MERA. In the box: a decoration of the indices of the MERA tensors with (red) arrows. Holographic screens are paths on the ambient manifold that extend between two boundary locations (e.g. P_1, P_2), intersect only copy tensors, and the red arrows in the immediate exterior of the path are all incoming to the interior. The reduced density matrix of the 2D interior (highlighted in yellow) transforms to the reduced density matrix of the bulk sites located on the (1D) screen under conjugation by an isometry, Eq. (4). (a) A geodesic holographic screen (green dashes). (b) A non-geodesic holographic screen (green dashes). (c) Example of a path (green dashes) that *does not* furnish a holographic screen, since some of the red arrows are outgoing from the interior.

[9]. Thus, the projection from a bulk region to an enclosing holographic screen may be viewed as the bulk dual of the action of a local conformal transformations on the boundary state.

V. A BULK/BOUNDARY DICTIONARY

In this section, we describe a simple prescription to obtain correlators and block Reyni entanglement entropy of a *critical* ground state $|\Psi^{\text{bound}}\rangle$ from a dual copy bulk state $|\Psi^{\text{bulk}}\rangle$. This is schematically depicted in Fig. 4. For simpler presentation, here we only list the formulae for calculating these boundary properties from the bulk, while their derivation is presented in Appendix E.

For the purpose of this section, we denote by x, x', x'' the locations of special sites of the boundary lattice \mathcal{L} , namely, x locates a site associated with an open index of the MERA at the base of an arbitrarily long vertical graph geodesic [see Fig. 2(a)], $|x - x''| = 3^q$, and $|x' - x''| = 3^{q'}$ where q, q' are positive integers. Here by ‘graph geodesic’ we mean the shortest connected path between two locations along the MERA graph itself, in contrast with geodesics on the ambient manifold outside the lattice that were considered in the previous section.

Consider the *one-site scaling superoperator* \hat{S} obtained from the MERA tensor \hat{w} as

$$(\hat{S}_i^k)_k^{k'} = \sum_{j,l} \hat{w}_{jkl} (\hat{w}^\dagger)_{i'}^{jk'l},$$

and let \hat{o} and λ denote an eigenoperator and the corresponding eigenvalue of \hat{S} , namely, $\hat{S}(\hat{o}) = \lambda\hat{o}$. Operator \hat{o} is identified with a scaling operator of the underlying CFT with scaling dimension $\Delta = -\log_3 \lambda$ [3]. Also, let $\mathcal{G}_{x,x'}$ denote the set of bulk sites that are located along the graph geodesic extending between the bulk sites at $(x, 0)$ and $(x', 0)$.

The *2-point boundary correlator* of scaling operators \hat{o}_α and \hat{o}_β , applied at site locations x and x' , can be obtained from the bulk state as

$$\langle \hat{o}_\alpha(x) \hat{o}_\beta(x') \rangle_{\text{bound}} = f(\hat{u}, \hat{w}, \hat{o}_\alpha, \hat{o}_\beta) \times \langle \hat{K}_{x,x'} \hat{o}_\alpha(x, 0) \hat{o}_\beta(x', 0) \rangle_{\text{bulk}}, \quad (7)$$

where function $f(\hat{u}, \hat{w}, \hat{o}_\alpha, \hat{o}_\beta)$ is defined in Appendix E, and $\hat{K}_{x,x'} = \bigotimes_i \hat{P}_i$ is a *string operator* that projects each bulk site $i \in \mathcal{G}_{x,x'}$ to the state $|+\rangle$ [Eq. (3)]. That is, the boundary correlator can be obtained by calculating the same correlator from the bulk state but after projecting all the bulk sites in $\mathcal{G}_{x,x'}$ to the state $|+\rangle$.

Analogously, the *3-point boundary correlator* of scaling operators $\hat{o}_\alpha, \hat{o}_\beta$ and \hat{o}_γ can be obtained from the bulk state as

$$\langle \hat{o}_\alpha(x) \hat{o}_\beta(x') \hat{o}_\gamma(x'') \rangle_{\text{bound}} = g(\hat{u}, \hat{w}, \hat{o}_\alpha, \hat{o}_\beta, \hat{o}_\gamma) \times \langle \hat{T}_{x,x',x''} \hat{o}_\alpha(x, 0) \hat{o}_\beta(x', 0) \hat{o}_\gamma(x'', 0) \rangle_{\text{bulk}}, \quad (8)$$

where function $g(\hat{u}, \hat{w}, \hat{o}_\alpha, \hat{o}_\beta, \hat{o}_\gamma)$ is defined in Appendix E, and $\hat{T}_{x,x',x''} = \bigotimes_i \hat{P}_i$ is a *branched string operator* that projects all bulk sites $i \in \{\mathcal{G}_{x,x'} \cup \mathcal{G}_{x',x''}\}$ to the state $|+\rangle$.

More generally, the n -point correlator $\langle \hat{o}_\alpha(x_1), \hat{o}_\beta(x_2), \dots, \hat{o}_\nu(x_n) \rangle_{\text{bound}}$ is proportional to

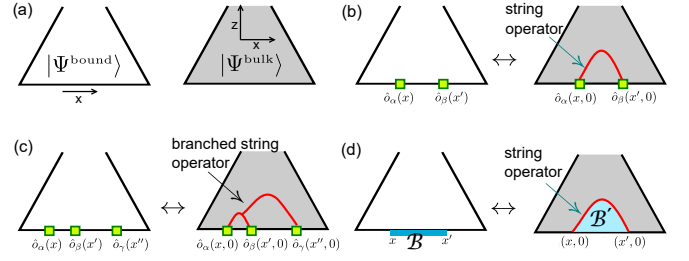


FIG. 4. (Color online) A dictionary that translates between properties of a critical ground state $|\Psi^{\text{bound}}\rangle$ and a dual copy bulk state $|\Psi^{\text{bulk}}\rangle$. Here x, x', x'' locate special sites on the lattice \mathcal{L} , see main text. (a) A schematic depiction of the MERA (represents $|\Psi^{\text{bound}}\rangle$) and a copy-lifted MERA (represents $|\Psi^{\text{bulk}}\rangle$). (b) The 2-point boundary correlator $\langle \hat{o}_\alpha(x) \hat{o}_\beta(x') \rangle_{\text{bound}}$ is proportional to the bulk expectation value of a string operator, Eq. (7). (c) The 3-point correlator $\langle \hat{o}_\alpha(x) \hat{o}_\beta(x') \hat{o}_\gamma(x'') \rangle_{\text{bound}}$ is proportional to the bulk expectation value of a branched string operator, Eq. (8). (d) The Reyni entropy of a boundary region $\mathcal{B} \subset \mathcal{L}$ is equal to the Reyni entropy (plus a constant) of a bulk region $\mathcal{B}' \subset \mathcal{M}$ as calculated from the projected bulk state $|\Omega^{\text{bulk}}\rangle$, Eq. (9). \mathcal{B}' is comprised of sites enclosed between the geodesic that extends between x and x' and the boundary, including the boundary sites of \mathcal{M} located at $(x+1, 0), (x+2, 0), \dots, (x'-1, 0)$.

the bulk expectation value of the n scaling operators tensor product with multi-branched string projection operator with $n-2$ branch points. (Here x_1 locates a site of \mathcal{L} that is associated with an open index of the MERA at the base of an arbitrarily long vertical graph geodesic, and $|x_i - x_j| = 3^{q_{ij}}$ for all i, j where q_{ij} is a positive integer.)

Thus, n -point correlators of an 1D critical ground state translate to the expectation value of *extended* operators in a dual 2D copy bulk state. This identification, though extremely simple, appears to caricature the prescription of perturbatively calculating boundary correlators in AdS/CFT by evaluating *Witten diagrams* [5, 14]. (The latter is a type of Feynman diagram that involves integrating certain bulk degrees of freedom and propagating the boundary operators into the bulk. Note that applying the projector \hat{P} , Eq. (3), on a bulk site in effect sums the basis vectors at the site.)

Note that the extended operators that appear in Eq. (7) and Eq. (8) act on a *finite* number of bulk sites. On the other hand, any of the boundary correlators considered above can be obtained *exactly* [namely, without the multiplicative factors $f(\hat{u}, \hat{w}, \hat{o}_\alpha, \hat{o}_\beta)$ and $g(\hat{u}, \hat{w}, \hat{o}_\alpha, \hat{o}_\beta, \hat{o}_\gamma)$] by computing the same correlator in the bulk state but after projecting an *infinite* number of bulk sites—those located in the joint causal cone of the sites on which the scaling operators act—to the state $|+\rangle$. (This can be seen by matching the tensor network contraction equating to the boundary correlator with the tensor network contraction equating to the same correlator in the bulk after projecting all the bulk sites in the joint causal cone.) The point here is that in order to ob-

tain the critical exponents, part of the underlying CFT data, from the bulk it suffices to consider the bulk expectation value of extended operators that act only on a finite number of bulk sites.

The Reyni entanglement entropy of a block B of sites in a quantum many-body state $|\Psi\rangle$ is:

$$R_\alpha(B) = -\log_2 \text{Tr} \hat{\rho}_B^\alpha, \quad \alpha = 2, 3, \dots,$$

where $\hat{\rho}_B$ is the reduced density matrix of the block B , obtained by tracing out all sites belonging to the complement of B in state $|\Psi\rangle$. The boundary Reyni entanglement entropy $R_\alpha^{\text{bound}}(\mathcal{B})$ of a block $\mathcal{B} \subset \mathcal{L}$ of sites located at $x, x+1, \dots, x'$ can be also be obtained from the bulk in a simple way in the limit of large block size $|x-x'|$. We have

$$R_\alpha^{\text{bound}}(\mathcal{B}) \approx R_\alpha^{\text{bulk}}(\mathcal{B}') + h(\hat{u}, \hat{w}), \quad \text{large } |x-x'|. \quad (9)$$

Here $\mathcal{B}' \subset \mathcal{M}$ [see Fig. 4(d)] is the set of bulk sites enclosed between the graph geodesic extending between locations $(x, 0)$ and $(x', 0)$ and the boundary, and including the bulk sites located at $(x+1, 0), (x+2, 0), \dots, (x'-1, 0)$. $R_\alpha^{\text{bulk}}(\mathcal{B}')$ is the Reyni entanglement entropy of \mathcal{B}' calculated from the *projected bulk state* $|\Omega^{\text{bulk}}\rangle = \hat{K}_{x,x'} |\Psi^{\text{bulk}}\rangle$, and $h(\hat{u}, \hat{w})$ is a known function of the MERA tensors defined in Appendix E. Note that Eq. (9) differs from the equality Eq. (6) between the entanglement entropy of sites located on and inside a holographic screen respectively. While Eq. (9) relates a *boundary* Reyni entanglement entropy to a *bulk* Reyni entanglement entropy and holds approximately in the limit of large $|x-x'|$, Eq. (6) is an exact property of a copy bulk state.

VI. BULK ENTANGLEMENT VS BOUNDARY CENTRAL CHARGE

Certain bulk features in the AdS/CFT correspondence depend on the central charge of the CFT. For example, correlations due to quantum fluctuations in the bulk are generally suppressed for large central charge [12]. Another example is the Ryu-Takayanagi formula, which holds when the bulk is described by classical gravity [17]. For instance, for an $1+1$ dimensional CFT that has a classical gravity dual, the Ryu-Takayanagi formula equates (in appropriate units) the von Neumann entanglement entropy S_ℓ^{bound} of an interval of length ℓ in the CFT vacuum to the length L_{geo} of the geodesic that extends between the end points of the interval through the dual bulk spacetime, $S_\ell^{\text{bound}} \approx L_{\text{geo}}$. (An analogous holographic interpretation of Reyni entanglement entropy has also been proposed [18].) On the other hand, we have $S_\ell^{\text{bound}} = \frac{c}{3} \log \ell$, where c is the central charge [19]. Thus, S_ℓ^{bound} , and therefore L_{geo} determined in the dual bulk geometry, increases with the central charge c .

In this paper, we have not derived a (semi-)classical bulk geometry from the boundary state. So instead

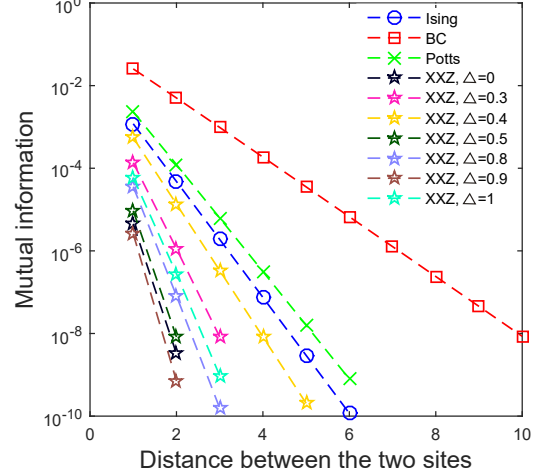


FIG. 5. (Color online) The mutual information $I_{z,z'}^{\text{bulk}}$ [Eq. (11)] in copy bulk states, dual to the ground state of each of the critical spin chains listed in Eq. (10), between two bulk sites located at (x, z) and (x, z') respectively (separated by distance $|z-z'|$) as illustrated in Fig. 11(c).

we probe for any potential dependence of the entanglement/correlations in copy bulk states, dual to critical boundary states, on the central charge. To this end, we considered the ground states of the following 1D critical spin models with different central charges:

$$\begin{aligned} \hat{H}^{\text{ISING}} &= \sum_i \hat{\sigma}_z^i \hat{\sigma}_z^{i+1} + \hat{\sigma}_x^i, \\ \hat{H}^{\text{BC}} &= \sum_i -\hat{X}^i \hat{X}^{i+1} + \alpha (\hat{X}^i)^2 + \beta (\hat{Z}^i)^2, \\ \hat{H}^{\text{POTTS}} &= -\sum_i \hat{P}^i (\hat{P}^T)^{i+1} + (\hat{P}^T)^i \hat{P}^{i+1} + \hat{M}^i, \\ \hat{H}^{\text{XXZ}} &= \sum_i \hat{\sigma}_x^i \hat{\sigma}_x^{i+1} + \hat{\sigma}_y^i \hat{\sigma}_y^{i+1} + \Delta \hat{\sigma}_z^i \hat{\sigma}_z^{i+1}, \end{aligned} \quad (10)$$

where i labels sites of an 1D lattice on which the Hamiltonian acts, $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are Pauli matrices, \hat{X}, \hat{Z} are spin-1 operators, and \hat{P}, \hat{M} are 3×3 Potts matrices:

$$\hat{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad \hat{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

\hat{H}^{ISING} has central charge $\frac{1}{2}$, \hat{H}^{BC} is the spin-1 Blume Capel model which is critical for $\alpha = 0.910207, \beta = 0.415685$ with central charge $\frac{7}{10}$, \hat{H}^{POTTS} has central charge $\frac{4}{5}$, and \hat{H}^{XXZ} is critical for $-1 \leq \Delta < 1$ with central charge 1. (The scaling dimensions of the CFT underlying \hat{H}^{XXZ} vary continuously with Δ .) For the results presented in this section, we considered values of $\Delta \in \{0, 0.3, 0.4, 0.5, 0.8, 0.9, 1\}$.

The MERA representation of the ground states of these models was obtained using the variational energy minimization algorithm [15] keeping bond dimension $\chi = 12$.

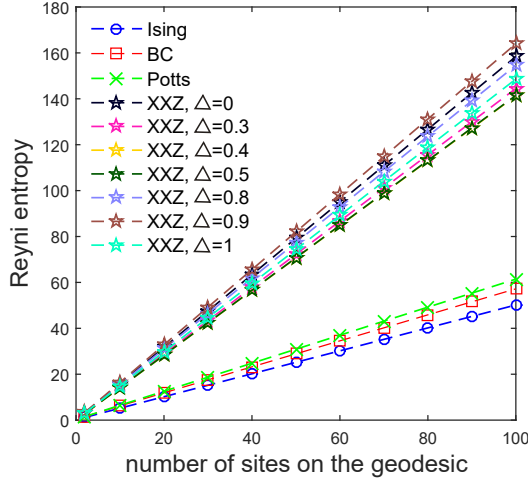


FIG. 6. (Color online) Second Reyni entanglement entropy in copy bulk states, dual to the ground state of each of the critical spin chains listed in Eq. (10), of bulk sites located on a geodesic holographic screen [such as the one illustrated in Fig. 11(c)].

In these simulations, the error in the ground state energy density for the Ising model was $O(10^{-8})$ while the relative error in the central charge was 0.4%. For the remaining models, the error in the ground state energy density was $O(10^{-5})$ and the relative error in the central charge was at most 1.2%. The relative error in the first few scaling dimensions for all the models was at most 4%.

The Hamiltonians listed in Eq. (10) are not scale-invariant, but flow to a scale-invariant fixed point after possibly several RG (entanglement renormalization) steps. We considered the renormalized scale-invariant ground state of each model, described by retaining only the scale-invariant part of the MERA tensor network [3]. We first translated the scale-invariant part of the MERA to a constrained MERA form based on higher order singular value decomposition, and then lifted the modified MERA by inserting copy tensors on the bonds to obtain a dual copy bulk state, as described in Appendix C. See discussion in Sec. III.

Let us parameterize the bulk lattice \mathcal{M} by coordinates (x, z) where z corresponds to the length scale and x labels spatial translations (in the boundary description). For the purposes of this section, let (x, z) , (x, z') and (x', z) locate bulk sites along an arbitrarily long vertical graph geodesic [illustrated in Fig. 2(a)] such that $z \neq z'$ and $|x - x'| = 3^p$ where p is a positive integer.

For the ground state of each critical Hamiltonian listed in Eq. (10), we estimated the correlation length in the dual copy bulk state along the z direction from the scaling of the mutual information $I_{z,z'}^{\text{bulk}}$ given by

$$I_{z,z'}^{\text{bulk}} = S_z^{\text{bulk}} + S_{z'}^{\text{bulk}} - S_{z,z'}^{\text{bulk}}. \quad (11)$$

Here $S_z^{\text{bulk}} = -\text{Tr}(\hat{\rho}_z^{\text{bulk}} \log_2 \hat{\rho}_z^{\text{bulk}})$ is the von Neumann entanglement entropy of the bulk site located at (x, z) , $S_{z'}^{\text{bulk}}$

is the von Neumann entanglement entropy of the bulk site located at (x, z') , and $S_{z,z'}^{\text{bulk}}$ is the joint von Neumann entanglement entropy of the two sites, see Fig. 11(c).

The results are plotted in Fig. 5. We find that the mutual information decays exponentially, which implies that the bulk state has a finite correlation length along this direction. (The mutual information gives an upper bound for all 2-point correlators.) The plot also suggests a trend that the correlation length generally decreases with increase in boundary central charge for these models (with the exception of the Ising model). On the other hand, as mentioned previously, correlations due to quantum fluctuations in the bulk are also generally suppressed for large central charge in the AdS/CFT correspondence.

We also computed the second Reyni entanglement entropy $R_{x,x'}^{\text{bulk}} = -\log_2 \text{Tr}(\hat{\rho}_{x,x'}^{\text{bulk}})^2$. Here $\hat{\rho}_{x,x'}^{\text{bulk}}$ is the reduced density matrix of the bulk sites that are located on a geodesic holographic screen that is anchored next to the bulk sites located at $(x, 0)$ and $(x', 0)$ [see Sec. IV and also Fig. 11(c)]. The results are plotted in Fig. 6. We find that the $R_{x,x'}^{\text{bulk}}$ increases linearly with the number of bulk sites located on the screen (proportional to p), which is consistent with the area law scaling of bulk entanglement derived in Appendix B. Interestingly, the slope of the scaling of $R_{x,x'}^{\text{bulk}}$ also appears to increase monotonically with the central charge for these models.

Furthermore, the value of Reyni entanglement entropy $R_{x,x'}^{\text{bulk}}$ for a given block size is larger for a spin chain with a larger central charge. On the other hand, as described at the beginning of this section, in AdS/CFT the geodesic length L_{geo} also increases with boundary central charge, in accordance with the Ryu-Takayanagi formula. In view of this, it may be interesting to explore whether a bulk metric can be deduced from the bulk entanglement entropy. For example, is the entanglement entropy of the bulk sites intersected by a geodesic, plotted in Fig. 6, a legitimate measure of the geodesic's length, $L_{\text{geo}} \approx R_{x,x'}^{\text{bulk}}$? We leave this as an open question for future work. (Computing the von Neumann entanglement entropy, which appears in the Ryu-Takayanagi formula, incurs a higher computational cost. But for short geodesics we verified that the slopes of the scaling of von Neumann entanglement entropy, analogous to $R_{x,x'}^{\text{bulk}}$, also increase monotonically with increase in central charge for these models.)

VII. SUMMARY AND OUTLOOK

We introduced a toy holographic correspondence for quantum lattice systems that is both motivated by and incorporates some general features of the AdS/CFT correspondence, listed (i)-(iv) in the introduction. To summarize, we lifted the MERA representation—which also describes the RG flow—of an 1D ground state to a tensor network representation of a dual 2D quantum state, such that the two states are seen to live on the boundary and in the bulk of a 2D manifold. We achieved this by embedding the MERA in a 2D manifold and inserting tensors

with open indices on the bonds of the MERA. The open indices of the MERA and the lifted MERA are associated with the boundary and the emergent bulk degrees of freedom respectively, and the lifted MERA represents the dual bulk state.

We explored parallels between this tensor network state correspondence and the AdS/CFT correspondence, in light of the ongoing dialogue between the MERA and holography. For example, we described how copy bulk states exhibit the presence of holographic screens, and also a simple prescription to obtain boundary correlators from expectation values of extended operators in the bulk. Both these properties remind of the AdS/CFT correspondence.

The bulk construction described in this paper can be generalized in several ways. Motivated by further guidelines from holography, one may fix the bond tensors differently, or pre-process the tensor network in some way before lifting it (e.g. see Appendix C). It may also be interesting to consider inserting bond tensors on a fewer number of bonds e.g. only those that are output from the \hat{w} tensors since, strictly speaking, only these are associated with renormalized sites in the MERA, see Fig. 1. This corresponds to a fewer number of bulk sites. On the other hand, allocating bulk sites to all the bonds, as we have done in this paper, allows the introduction of suitable gauge transformations in the bulk, which are dual to

a global on-site symmetry at the boundary [11]. In the presence of on-site symmetries at the boundary, it is also more natural to associate two bulk sites with every bond of the MERA, which corresponds to lifting the MERA by inserting a 4-index bond tensor with each index taking χ values. (Or equivalently, a 3-index bond tensor, as in this paper, but whose open index takes χ^2 values.)

In conclusion, we have tried to make a case for obtaining an emergent 2D description from the MERA representation of an 1D ground state by associating dual degrees of freedom with the bonds of the tensor network. The formalism introduced in this paper illustrates a possible way in which the MERA could implement holography, and more generally how a tensor network with open indices intrinsically encodes a correspondence between two quantum many-body states (after a lifting action is defined for the bond indices).

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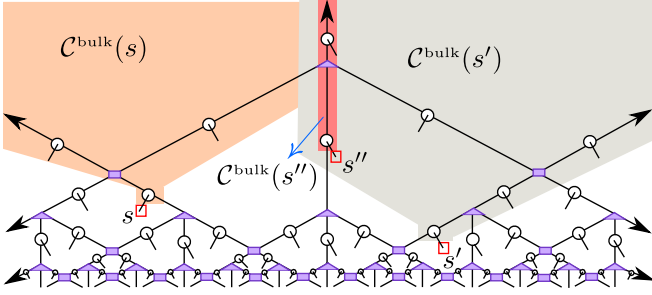


FIG. 7. (Color online) Causal cone structure of the lifted MERA. The reduced density matrix of a bulk site $s \in \mathcal{M}$ depends only a subset $\mathcal{C}^{\text{bulk}}(s)$ of the lifted MERA tensors. $\mathcal{C}^{\text{bulk}}(s)$ is the causal cone of site s . In particular, $\mathcal{C}^{\text{bulk}}(s)$ comprises tensors which only appear at length scales larger than that of s , and each length scale contributes only a bounded number of tensors. Shown here are the causal cones of three different bulk sites $s, s', s'' \in \mathcal{M}$. The reduced density matrix of multiple bulk sites depends only on the tensors that belong to the union of the causal cones of the individual sites.

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Appendix A: Causal cone structure of a copy lifted MERA

Analogous to the MERA, a copy-lifted MERA is also endowed with a bounded-width causal cone structure see Fig. 7. The causal cone structure of the copy-lifted MERA results from the fact that, in addition to tensors \hat{u} and \hat{w} being isometric [Eq. (1)], the copy tensor is also isometric, namely,

$$\sum_{qr} \hat{c}_{qr}^p (\hat{c}^*)_{p'}^{qr} = \delta_{p'}^p, \quad (\hat{c}^*)_{p'}^{qr} = \hat{c}_{qr}^p. \quad (\text{A1})$$

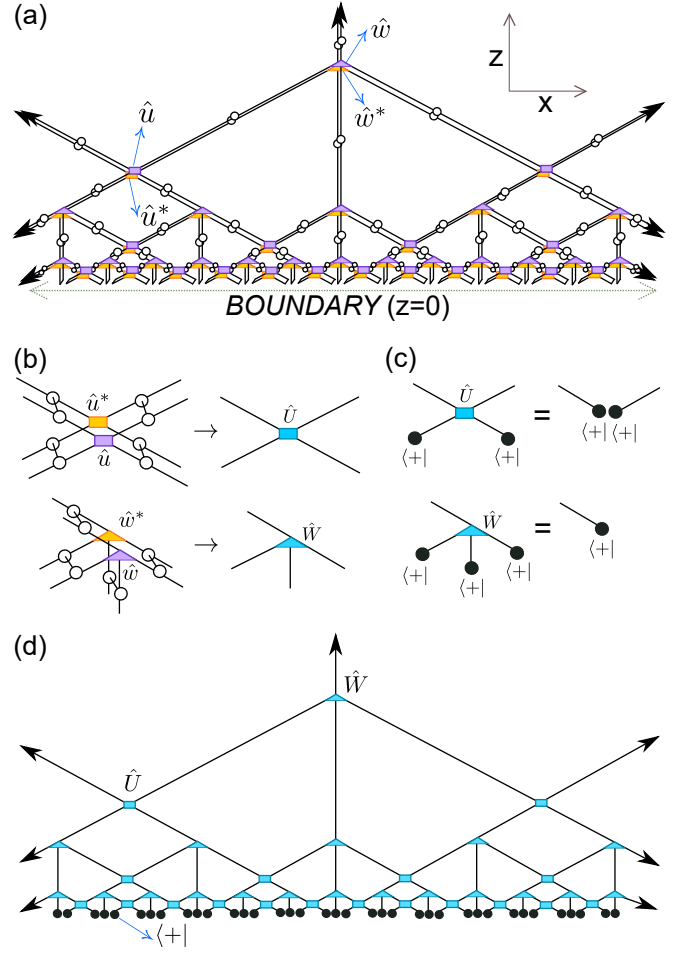


FIG. 8. (Color online) (a) Tensor network contraction equating to the norm of the bulk state. The open indices of the lifted MERA, including the indices located at the boundary $z = 0$, are contracted with the respective open indices in the conjugate lifted MERA (obtained by replacing each tensor of the lifted MERA by its conjugate). (b) Stochastic tensor \hat{U} is obtained by contracting tensor \hat{u} , its complex conjugate \hat{u}^* , and 8 copy tensors as shown. Stochastic tensor \hat{W} is obtained by contracting tensor \hat{w} , its complex conjugate \hat{w}^* , and 8 copy tensors as shown. (c) Equalities satisfied by the stochastic tensors \hat{U} and \hat{W} . (d) The tensor network contraction shown in (a) equates to a contraction of a tensor network made of copies of stochastic tensors \hat{U}, \hat{W} and the vector $\langle + |$.

(In fact, since the copy tensor is invariant under any permutation of its indices, it is an isometry after any two of its indices are grouped into a single index.) The causal cone structure aids in the efficient computation of expectation values from a copy-lifted MERA.

Consider the tensor network contraction that equates to the norm of a copy bulk state, depicted in Fig. 8(a). The contraction reduces to a contraction of a tensor network that is made of stochastic tensors and is closed with copies of the vector $\langle + |$ at $z = 0$, as shown in Fig. 8(b)-(c). The contraction of the stochastic tensor network

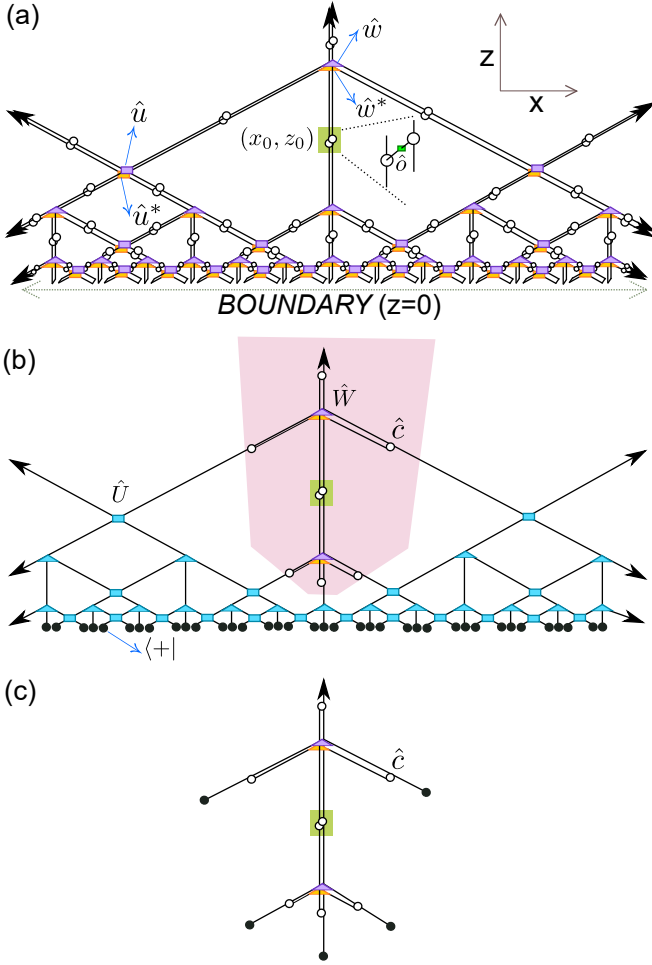


FIG. 9. (Color online) (a) Tensor network contraction equating to the bulk expectation value of a one site operator \hat{o} located at (x_0, z_0) . The contraction reduced to the contraction shown in (b), which consists of copies of the stochastic tensor \hat{U} and \hat{W} and some of the original MERA tensors. (c) The contraction shown in (b) simplifies to a contraction of tensors only in the bulk causal cone [highlighted in (b)] of the site located at (x_0, z_0) .

simplifies since

$$\begin{aligned}\hat{U}(|+\rangle \otimes |+\rangle) &= (|+\rangle \otimes |+\rangle), \\ \hat{W}(|+\rangle \otimes |+\rangle \otimes |+\rangle) &= |+\rangle,\end{aligned}\quad (\text{A2})$$

depicted in Fig. 8(d). Thus, the norm is equal to 1.

Next, consider the tensor network contraction that equates to the expectation value of an one-site operator \hat{o} acting at location (x_0, z_0) , as illustrated in Fig. 9(a). The tensors at the boundary $z = 0$ that are contracted with their Hermitian adjoints cancel out, which brings the tensors at the next length scale in contact with their Hermitian adjoints and so on. Thus, a cascade of pairwise contractions cancels all tensors located between $0 \leq z < z_0$. Furthermore, all tensors located at $z \geq z_0$ that appear outside the causal cone of the site (x_0, z_0) also cancel out. Ultimately, only tensors within the causal cone of

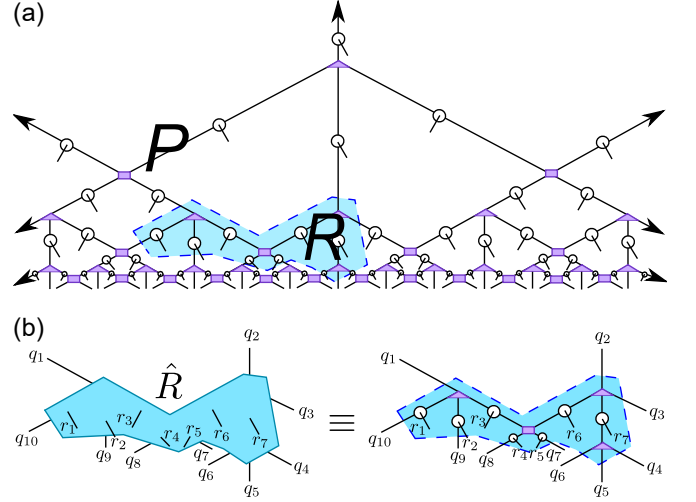


FIG. 10. (Color online) (a) A bipartition (highlighted in blue) of the lifted MERA into subnetworks P and R . (b) Tensor \hat{R} obtained by contracting all the tensors in R . Indices q_1, q_2, \dots, q_{10} connect R with P . r_1, r_2, \dots, r_7 are the open indices located inside R .

the site (x_0, z_0) remain, as depicted in Fig. 9(d). The number of tensors that appear in the causal cone at increasingly large length scales is bounded and therefore the tensors within the causal cone can be contracted together efficiently and exactly by using a transfer matrix approach (a technique commonly applied in MERA algorithms [15]).

Appendix B: Area law entanglement in the bulk

Define the perimeter (area of the boundary) of a 2D subsystem of the bulk lattice \mathcal{M} as the number of sites that are located at the subsystem's boundary. In this section, we prove that: *the subsystem entanglement entropy of the bulk state described by a lifted MERA scales at most as the perimeter of the subsystem*. Such a scaling is called ‘(boundary) area law entanglement’ in condensed matter physics, where it is commonly exhibited by ground states of local 1D and 2D Hamiltonians.

Proof.—Consider a bipartition of the lifted MERA tensor network into subnetworks P and R , see Fig. 10(a). This also corresponds to a bipartition of the bulk lattice \mathcal{M} into parts $\mathcal{P} \subset \mathcal{M}$ and $\mathcal{R} \subset \mathcal{M}$ comprised of sites associated with the (open indices of) copy tensors in subnetworks P and R respectively. The bulk state $|\Psi^{\text{bulk}}\rangle$ can be expressed as

$$|\Psi^{\text{bulk}}\rangle = \sum_{pqr} \hat{P}_{pq} \hat{R}_{qr} |p\rangle \otimes |q\rangle \otimes |r\rangle, \quad (\text{B1})$$

where $p \equiv (p_1, p_2, \dots)$, $q \equiv (q_1, q_2, \dots)$, $r \equiv (r_1, r_2, \dots)$, and \hat{P}_{pq} and \hat{R}_{qr} are obtained by contracting all tensors in subnetworks P and R respectively, see Fig. 10(b). The

reduced density matrix $\hat{\rho}^{\mathcal{R}}$ of \mathcal{R} is

$$\hat{\rho}^{\mathcal{R}} = \sum_{rr'} \sum_q \sum_p \hat{P}_{pq} P_{pq}^* \hat{R}_{qr} \hat{R}_{qr'}^* |r\rangle\langle r'|, \quad (\text{B2})$$

which may be re-expressed as

$$\hat{\rho}^{\mathcal{R}} = \sum_{rr'} \left(\sum_q \hat{S}_{qr} \hat{R}_{qr'}^* \right) |r\rangle\langle r'|, \quad (\text{B3})$$

where $\hat{S}_{qr} = \sum_p \hat{P}_{pq} \hat{P}_{pq}^* \hat{R}_{qr}$. We can write Eq. (B3) in matrix form as $\hat{\rho}^{\mathcal{R}} = \hat{S}^T \hat{R}^*$, where matrices \hat{S}^T and \hat{R}^* have dimensions $|r| \times |q|$ and $|q| \times |r|$ respectively. Here $|i|$ denotes the number of values that index i takes. This implies that if $|r| \geq |q|$ then the rank of $\hat{\rho}^{\mathcal{R}}$ is less than equal to $|q|$. This, in turn, implies that the entanglement entropy $S(\hat{\rho}^{\mathcal{R}}) = -\text{Tr}(\hat{\rho}^{\mathcal{R}} \log_2 \hat{\rho}^{\mathcal{R}})$ of a sufficiently large subsystem \mathcal{R} is $\mathcal{O}(\log |q|)$, where $\log |q|$ is the number of sites that comprise the boundary of \mathcal{R} . Thus, the bulk subsystem entanglement entropy scales at most as the perimeter of the subsystem. ■

Note that the proof does not depend on the components of the tensors, but only on the fact that the bulk state is encoded in a lifted MERA tensor network. We remark that (boundary) area law entanglement scaling proved above is subtle in a 2D hyperbolic geometry (here the hyperbolic lattice \mathcal{M}) where the area of a subsystem can be proportional to its perimeter.

Appendix C: Constrained MERA representations

In this appendix, we describe how to translate the standard MERA representation, reviewed in Sec. II, to a MERA representation in which the intrinsic bond freedom is significantly constrained. See discussion in Sec. III A. The plots in Fig. 5 and Fig. 6 were obtained by using the constrained MERA representation described next.

1. Higher order singular value decomposition

Define the norm of a tensor $\hat{T}_{i_1 i_2 \dots i_n}$ as

$$\|\hat{T}\| = \sum_{i_1 i_2 \dots i_n} \hat{T}_{i_1 i_2 \dots i_n} \hat{T}_{i_1 i_2 \dots i_n}^*. \quad (\text{C1})$$

Any tensor $\hat{T}_{i_1 i_2 \dots i_n}$ can be decomposed into a ‘core tensor’ $\hat{S}_{i_1 i_2 \dots i_n}$ and a set of n unitary matrices $\hat{U}^{[k]}$ ($k \in \{1, 2, \dots, n\}$), one for each index of \hat{T} such that

$$\hat{T}_{i_1 i_2 \dots i_n} = \sum_{i'_1 i'_2 \dots i'_n} \hat{S}_{i'_1 i'_2 \dots i'_n} (\hat{U}^{[1]})_{i_1}^{i'_1} (\hat{U}^{[2]})_{i_2}^{i'_2} \dots (\hat{U}^{[n]})_{i_n}^{i'_n}, \quad (\text{C2})$$

and the core tensor \hat{S} satisfies

$$\|\hat{S}^{[m,p]}\| \geq \|\hat{S}^{[m,p']}\|, \quad \text{if } p > p', \quad \forall m. \quad (\text{C3})$$

Here $\hat{S}^{[m,p]}$ is a $(n-1)$ -index tensor obtained from \hat{S} by fixing the m^{th} index of \hat{S} to value p , that is,

$$(\hat{S}^{[m,p]})_{j_1 j_2 j_{n-1}} \equiv \hat{S}_{j_1 \dots j_m=p \dots j_{n-1}}. \quad (\text{C4})$$

Equation C2, along with the constraint Eq. (C3), is known as the *higher order singular value decomposition* (HOSVD). It can be obtained by means of a sequence of matrix singular value decompositions [20]. Note that if tensor \hat{T} can be reshaped into an isometric matrix according to some bipartition of its indices then the core tensor \hat{S} is also an isometry according to the same bipartition. Also, if \hat{T} is a unitary tensor (according to some bipartition of its indices) it can be easily shown that $\|\hat{T}^{[m,p]}\|$ is constant for all m, p . This means that given an HOSVD decomposition of a unitary tensor one can contract the core tensor \hat{S} with an *arbitrary* unitary matrix \hat{R} on any index k and update $\hat{U}^{[k]'} = \hat{R}^\dagger \hat{U}^{[k]}$ to obtain another HOSVD of the tensor. On the other hand, the most constrained HOSVD possible for a generic tensor is unique up to contractions with only *diagonal* unitary matrices.

Our goal here is to introduce a MERA representation that has a more constrained bond freedom than the standard MERA representation. Let us define a MERA representation by requiring that each tensor of the MERA

1. is unitary or isometric, and
2. satisfies Eq. (C3).

We refer to this MERA representation as an *HOSVD-MERA*. Given a MERA representation, decompose each tensor according to Eq. (C2) and subsequently multiply together the two unitary matrices that appear on each bond. The resulting tensor network, an HOSVD-MERA, is comprised of core tensors (which replace the original tensors) and a set of unitary matrices, one for each bond.

However, as mentioned previously, if the given MERA contains unitary tensors the corresponding HOSVD-MERA representation is not any more constrained. On the other hand, HOSVD is likely to be more constrained for a tensor that is isometric (according to some bipartition of indices) but is not unitary according to *any* bipartition of indices. That is, the contraction of the core tensor in the HOSVD of a generic isometric tensor with an arbitrary unitary matrix \hat{R} generally breaks the constraint Eq. (C3). We can exploit this fact by contracting pairs of MERA tensors, \hat{w} and \hat{u} , to obtain a coarse-grained MERA made of only isometric tensors as depicted in Fig. 11(a). We then translate the coarse-grained MERA to an HOSVD-MERA representation as described above, namely, by applying HOSVD for each coarse-grained tensor, see Fig. 11(b)-(c).

The bond freedom in an HOSVD-MERA representation of a quantum state is more constrained than the standard MERA representation since its tensors are unique up to contractions only with diagonal unitary matrices. Subsequently, the different copy bulk states obtained by lifting the HOSVD-MERA, after applying different *diagonal* unitary bond transformations, are related

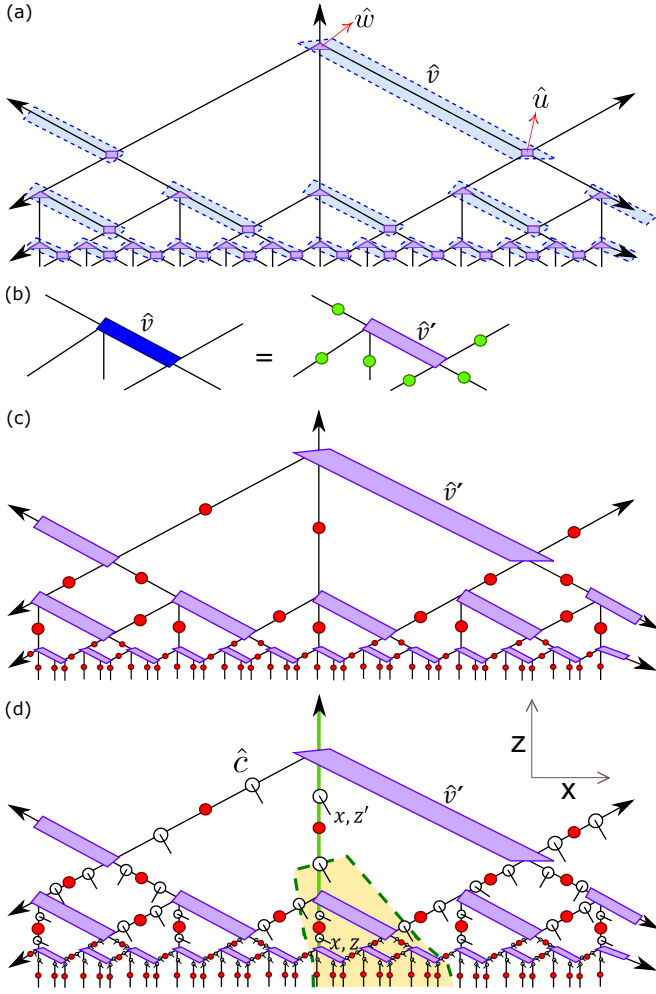


FIG. 11. (Color online) (a) Coarse-graining the MERA by contracting pairs of tensors \hat{u} and \hat{w} to obtain an isometric tensor \hat{v} . (b) Higher order singular value decomposition, Eq. (C2), of the coarse-grained isometric tensor \hat{v} into an isometric core tensor \hat{v}' (purple) and a unitary matrix (yellow) on each bond. (c) HOSVD-MERA obtained by decomposing each \hat{v} as shown in (b) and multiplying the two green unitary matrices that appear on each bond to obtain another unitary matrix (red). (d) Lifting the HOSVD-MERA by inserting copy tensors on all its bond indices. Fig. 5 plots the Reyni entanglement entropy of bulk sites located on a geodesic holographic screen, illustrated here as the green dashed path. Fig. 6 plots the mutual information between two bulk sites located at different length scales e.g. the bulk sites located at (x, z) and (x, z') along the vertical geodesic (solid green).

to one another by the action of one-site diagonal unitary transformations on the bulk lattice. This is because the copy tensor commutes with a diagonal matrix, namely, a contraction of the copy with a diagonal matrix on any index is equal to a contraction of the copy with the same diagonal matrix on a different index. Thus, in this case, the different copy bulk states, dual to the same boundary state, (at least) have the same entanglement.

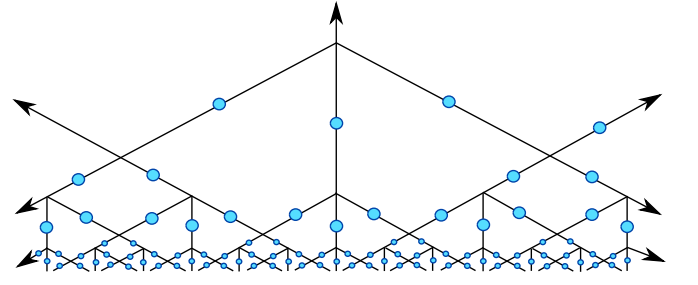


FIG. 12. (Color online) A matrix form of the MERA obtained by applying tensor rank decomposition for each tensor of the MERA. All the information of the quantum state represented by the MERA is collected into a set of matrices (blue), one for each bond of the MERA. Each vertex of the tensor network depicts a 4-index copy tensor that connects four matrices.

2. A matrix form of the MERA

Another constrained MERA representation can be obtained by means of the so called *tensor rank decomposition* of the MERA tensors. Tensor rank decomposition is an HOSVD where the core tensor is fixed to be the copy tensor [21]. (The components of an n -index copy tensor are 1 for equal values of the n indices and 0 otherwise.) For a generic tensor, tensor rank decomposition is generally unique up to systematic permutations of the bond matrices. Given a MERA representation, let us apply tensor rank decomposition for each tensor of the MERA and subsequently multiply together the two matrices that appear on each bond, see Fig. 12. In the resulting tensor network representation, all the information of the quantum state is captured in the bond matrices (since all remaining tensors are fixed as copy tensors). We refer to this representation as the *matrix form* of the MERA.

Note that a given MERA cannot be transformed into the matrix form by inserting resolutions of identity on the bonds, that is, by choosing a particular basis for the tensors. If the original MERA has bond dimension χ , the matrix form may have a bond dimension up to $\chi' = \chi^3$. We remark that finding the *optimal* tensor rank decomposition (corresponding to the smallest possible χ') of a generic tensor is NP-hard [21]. However, several algorithms are available to determine a non-optimal tensor rank decomposition of a generic tensor.

Appendix D: Proof of Eq. (4) for holographic screens

Let R denote the set of tensors located in the interior of a holographic screen, see Fig. 13(a). Denote by \hat{R} the tensor obtained by contracting together all tensors in R . Using components we write

$$\hat{R} \equiv \sum_{qr} \hat{R}_{qr} |q\rangle \langle r|, \quad (\text{D1})$$

where r is the tuple of all open indices in the interior and q is the tuple of all bond indices that connect the interior

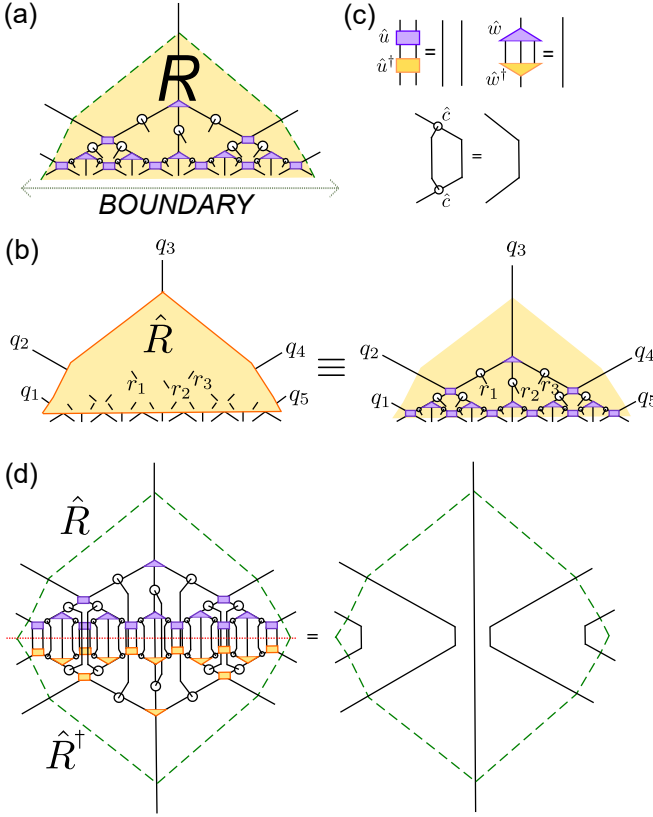


FIG. 13. (Color online) (a) A separate view of the interior R of the holographic screen depicted in Fig. 3. (b) Tensor \hat{R} that is obtained by contracting all tensors in the interior of the holographic screen, Eq. (D1). (c) Equalities satisfied by the isometric tensors \hat{u} , \hat{w} and the copy tensor \hat{c} . (d) Contraction of \hat{R} and its adjoint \hat{R}^\dagger is equal to the identity, Eq. (D2). (Thus, \hat{R} is an isometry.) This equality is obtained by applying the equalities depicted in (b) to pairwise contractions of tensors on the left hand side.

with the remaining tensor network, see Fig. 13(b). Because the holographic screen satisfies the arrow criterion stated in Sec. IV, it does not simultaneously intersect both an incoming and an outgoing index of any tensor of the lifted MERA. This means that tensor \hat{R} is simply a composition of copies of the isometric tensors \hat{u} , \hat{w} , and \hat{c} [Fig. 13(c)], and therefore \hat{R} itself is also an isometry satisfying

$$\hat{R}\hat{R}^\dagger = \hat{I}, \quad \sum_r \hat{R}_{qr} \hat{R}_{q'r}^* = \delta_{qq'}, \quad (\text{D2})$$

as depicted in Fig. 13(d).

Let us expand the bulk state according to Eq. (B1). The reduced density matrix $\hat{\rho}_{\text{geo}}^{\text{screen}}$ of the bulk sites located on the screen is

$$\hat{\rho}_{\text{geo}}^{\text{screen}} = \sum_{qq'} \left(\sum_p \hat{P}_{pq} \hat{P}_{pq'}^* \right) \left(\sum_r \hat{R}_{qr} \hat{R}_{q'r}^* \right) |q\rangle \langle q'|, \quad (\text{D3})$$

where p is the tuple of all open indices in the exterior and tensor \hat{P} is obtained by contracting together all tensors

in the exterior subnetwork. Using Eq. (D2), Eq. (D3) simplifies to

$$\hat{\rho}_{\text{geo}}^{\text{screen}} = \sum_{qq'} \left(\sum_p \hat{P}_{pq} \hat{P}_{pq'}^* \right) \delta_{qq'} |q\rangle \langle q'| = \sum_{qp} \hat{P}_{pq} \hat{P}_{pq}^* |q\rangle \langle q|. \quad (\text{D4})$$

Next, the reduced density matrix $\hat{\rho}_{\text{geo}}^{\text{interior}}$ of all the bulk sites in the interior is

$$\hat{\rho}_{\text{geo}}^{\text{interior}} = \sum_{rr'} \sum_q \sum_p \hat{P}_{pq} \hat{P}_{pq}^* \hat{R}_{qr} \hat{R}_{q'r'}^* |r\rangle \langle r'|, \quad (\text{D5})$$

which we re-arrange slightly as

$$\hat{\rho}_{\text{geo}}^{\text{interior}} = \sum_q \left(\sum_p \hat{P}_{pq} \hat{P}_{pq}^* \right) \left(\sum_r \hat{R}_{qr} |r\rangle \right) \left(\sum_{r'} \hat{R}_{q'r'}^* \langle r'| \right). \quad (\text{D6})$$

Multiplying $|r\rangle$ on both sides of Eq. (D1) we obtain

$$\hat{R}|r\rangle = \sum_q \hat{R}_{qr} |q\rangle. \quad (\text{D7})$$

Using Eq. (D7) and Eq. (D4) in Eq. (D6) we have

$$\begin{aligned} \hat{\rho}_{\text{geo}}^{\text{interior}} &= \hat{R}^\dagger \left(\sum_{qp} \hat{P}_{pq} \hat{P}_{pq}^* |q\rangle \langle q| \right) \hat{R} \\ &= \hat{R}^\dagger (\hat{\rho}_{\text{geo}}^{\text{screen}}) \hat{R}, \end{aligned} \quad (\text{D8})$$

which proves Eq. (4). ■

Note that the above proof does not rely on the actual components of the tensors. Thus, the presence of holographic screens is a general structural property of a copy lifted MERA.

Appendix E: Proof of the bulk/boundary dictionary

In this appendix, we prove the formulae Eqs. (7)-(9) that were listed in Sec. V of the main text. The proofs essentially employ two properties: (i) the causal cone structure of the lifted MERA (illustrated in Fig. 7), and (ii) the fact that the boundary state can be recovered from a copy bulk state by projecting each bulk site to the state $|+\rangle = \sum_{j=1}^X |j\rangle$, see Eq. (3). We prove each formula by illustrating that the tensor network contractions equating to the left hand side and the right hand side of the formula, are equal (up to a multiplicative or additive constant). [Note. – The notation and symbols used in this Appendix are introduced in Sec. V.]

1. 2-point correlators

The 2-point boundary correlator $\langle \hat{o}_\alpha(x) \hat{o}_\beta(x') \rangle_{\text{bound}}$ of scaling operators \hat{o}_α and \hat{o}_β acting at locations x and x' , $|x - x'| = 3^q$ (q is a positive integer), is obtained by a tensor network contraction illustrated on the left hand side in Fig. 14. Since tensors \hat{u} and \hat{w} are isometric, all tensors that are multiplied with their Hermitian

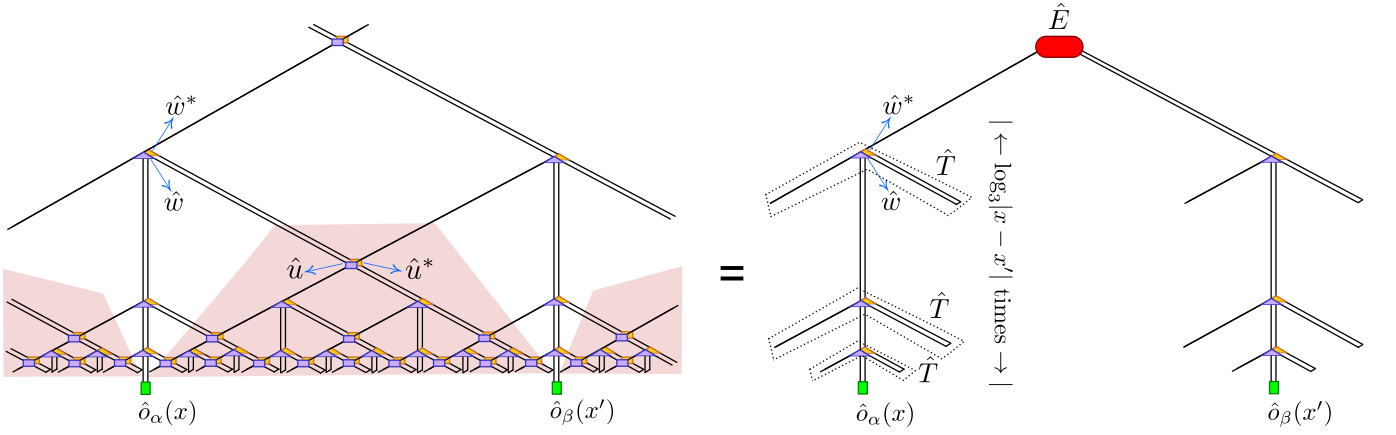


FIG. 14. (Color online) (Left) Tensor network contraction equating to the two point correlator $\langle \hat{o}_\alpha(x) \hat{o}_\beta(x') \rangle_{\text{bound}}$ of one-site operators \hat{o}_α and \hat{o}_β . The contraction consists of gluing the open indices of the MERA with the corresponding open indices of the conjugate MERA (namely, the tensor network obtained by replacing each tensor of MERA with its complex conjugate, shown in yellow), and sandwiching operators \hat{o}_α and \hat{o}_β on the open indices located at x and x' respectively. Tensors located inside the highlighted red regions are pairwise multiplied with their Hermitian adjoints and cancel out. The resulting contraction (shown on the right) consists of $p \equiv \log_3(|x - x'|)$ powers of a transfer superoperator \hat{T} (obtained by contracting tensors \hat{w} and \hat{w}^* enclosed within a dotted contour) and the environment tensor \hat{E} that is obtained by contracting all remaining tensors appearing at length scales larger than p .

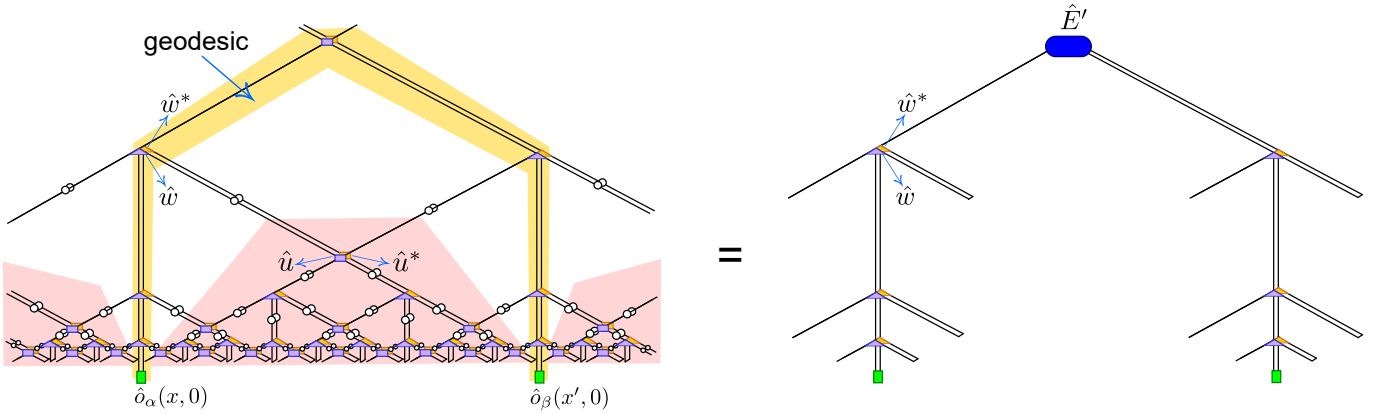


FIG. 15. (Color online) (Left) Tensor network contraction equating to the bulk expectation value of the string operator $\hat{K}_{x,x'} \hat{o}_\alpha(x,0) \hat{o}_\beta(x',0)$. The contraction consists of gluing the open indices of the lifted MERA with the corresponding open indices of the conjugate lifted MERA, sandwiching operators \hat{o}_α and \hat{o}_β on the open indices located at x and x' respectively, and sandwiching the operator $\hat{K}_{x,x'}$ on the open indices located along the graph geodesic (highlighted yellow) extending between $(x,0)$ and $(x',0)$. $\hat{K}_{x,x'}$ acts to remove the copy tensors on the geodesic. (Note that the copy tensors inside the yellow region have been removed.) Tensors located inside the highlighted red regions are pairwise multiplied with their Hermitian adjoints and cancel out. The resulting contraction (shown on the right) consists of $p \equiv \log_3(|x - x'|)$ powers of the same transfer superoperator \hat{T} that appears in Fig. 14 but a different environment tensor \hat{E}' , which is obtained by contracting all remaining tensors of the lifted MERA appearing at length scales larger than p .

adjoints cancel and the contraction simplifies to the one shown on the right hand side in Fig. 14. Here \hat{E} is the environment tensor obtained by contracting together all tensors at length scales above q that do not cancel out (that is, tensors located inside the causal cone). Using the fact that operators \hat{o}_α and \hat{o}_β are eigenoperators of the scaling superoperator \hat{T} with eigenvalues λ_α and λ_β

respectively, we obtain the closed expression

$$\langle \hat{o}_\alpha(x) \hat{o}_\beta(x') \rangle_{\text{bound}} = \frac{C_{\alpha\beta}}{|x - x'|^{\Delta_\alpha + \Delta_\beta}}, \quad (\text{E1})$$

where $C_{\alpha\beta} = \text{Tr}[\hat{E}(\hat{o}_\alpha \otimes \hat{o}_\beta)]$, and $\Delta_\alpha = -\log_3 \lambda_\alpha$ and $\Delta_\beta = -\log_3 \lambda_\beta$ are the scaling dimensions of the scaling operators \hat{o}_α and \hat{o}_β respectively.

Next, the corresponding bulk expectation value $\langle \hat{K}_{x,x'} \hat{o}_\alpha(x,0) \hat{o}_\beta(x',0) \rangle_{\text{bulk}}$ is obtained by contracting the

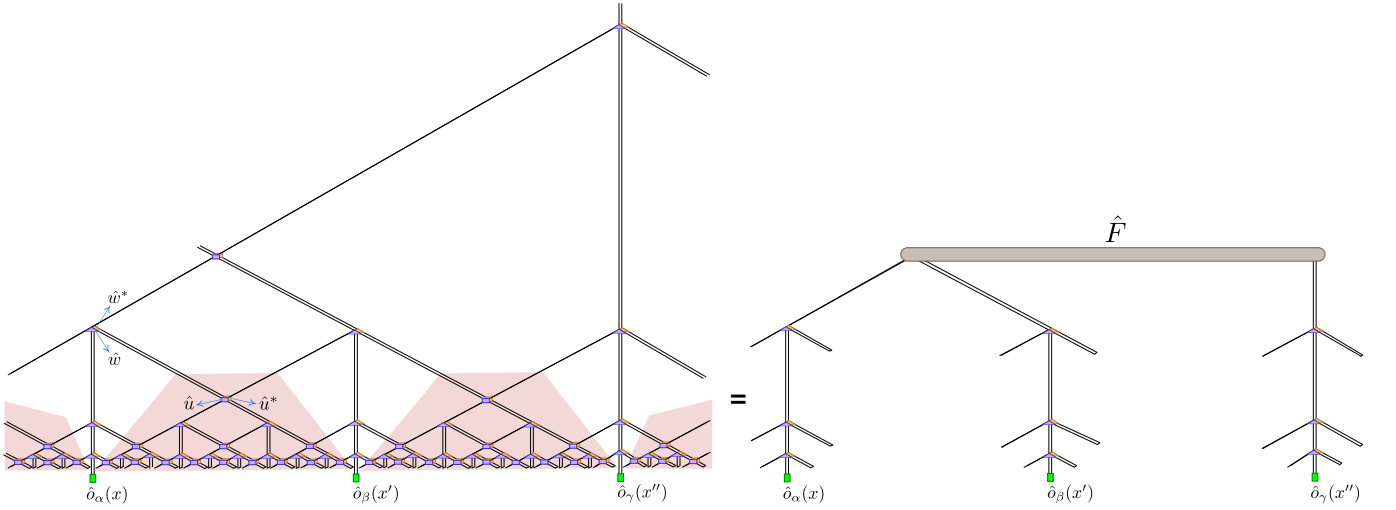


FIG. 16. (Color online) (Left) Tensor network contraction equating to the three point correlator $\langle \hat{o}_\alpha(x) \hat{o}_\beta(x') \hat{o}_\gamma(x'') \rangle_{\text{bound}}$ where e.g. $|x - x'| = |x' - x''| = \ell = 27$. The contraction consists of gluing the open indices of the MERA with the corresponding open indices of the conjugate MERA, and sandwiching operators $\hat{o}_\alpha, \hat{o}_\beta$ and \hat{o}_γ on the open indices located at x, x' and x'' respectively. Tensors located inside the highlighted red regions are pairwise multiplied with their Hermitian adjoints and cancel out. The resulting contraction (shown on the right) consists of $\log_3 \ell$ powers of the same transfer superoperator \hat{T} that appears in Fig. 14 and an environment tensor \hat{F} , which is obtained by contracting all remaining tensors appearing at length scales larger than $\log_3 \ell$.

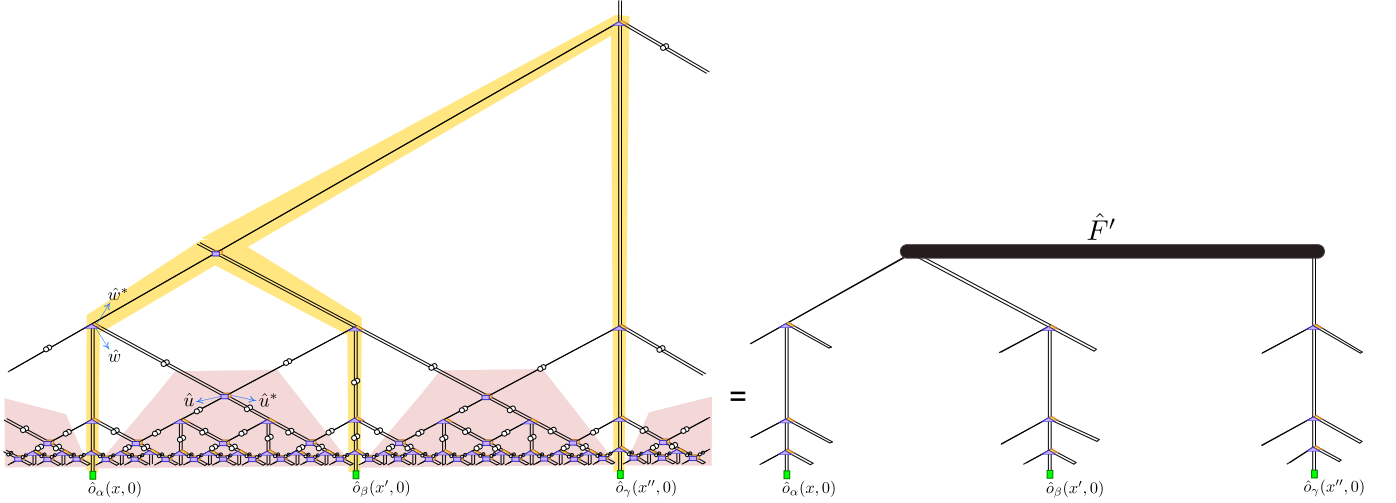


FIG. 17. (Color online) (Left) Tensor network contraction equating to the bulk expectation value of the extended operator $\hat{T}_{x,x',x''} \hat{o}_\alpha(x, 0) \hat{o}_\beta(x', 0) \hat{o}_\gamma(x'', 0)$ where e.g. $|x - x'| = |x' - x''| = \ell = 27$. The contraction consists of gluing the open indices of the lifted MERA with the corresponding open indices of the conjugate lifted MERA, sandwiching operators \hat{o}_α and \hat{o}_β on the open indices located at x and x' respectively, and sandwiching the operator $\hat{T}_{x,x',x''}$ on the open indices located on the union (highlighted yellow) of the graph geodesic between x and x' and the graph geodesic between x and x'' respectively. $\hat{T}_{x,x',x''}$ acts to remove the copy tensors within the highlighted yellow region. Tensors located inside the red highlighted regions are pairwise multiplied with their Hermitian adjoints and cancel out. The resulting contraction (shown on the right) consists of $\log_3 \ell$ powers of the same transfer superoperator \hat{T} that appears in Fig. 16 but a different environment tensor \hat{F}' , which is obtained by contracting all remaining tensors appearing at length scales larger than $\log_3 \ell$.

tensor network depicted on the left hand side in Fig. 15. Using the fact that tensors \hat{u} and \hat{w} are isometric, the contraction simplifies to the one shown on the right hand

side in Fig. 15, and we obtain the expression

$$\langle \hat{K}_{x,x'} \hat{o}_\alpha(x, 0) \hat{o}_\beta(x', 0) \rangle_{\text{bulk}} = \frac{C'_{\alpha\beta}}{|x - x'|^{\Delta_\alpha + \Delta_\beta}}, \quad (\text{E2})$$

where $C'_{\alpha\beta} = \text{Tr}[\hat{E}'(\hat{o}_\alpha \otimes \hat{o}_\beta)]$. Comparing Eq. (E1) and

Eq. (E2), we have

$$\langle \hat{\sigma}_\alpha(x) \hat{\sigma}_\beta(x') \rangle_{\text{bound}} = f(\hat{u}, \hat{w}, \hat{\sigma}_\alpha, \hat{\sigma}_\beta) \times \langle \hat{K}_{x,x'} \hat{\sigma}_\alpha(x, 0) \hat{\sigma}_\beta(x', 0) \rangle_{\text{bulk}}, \quad (\text{E3})$$

where $f(\hat{u}, \hat{w}, \hat{\sigma}_\alpha, \hat{\sigma}_\beta) = C_{\alpha\beta}/C'_{\alpha\beta}$, which is Eq. (7).

2. 3-point correlators

The 3-point boundary correlator $\langle \hat{\sigma}_\alpha(x) \hat{\sigma}_\beta(x') \hat{\sigma}_\gamma(x'') \rangle_{\text{bound}}$ is obtained by contracting a tensor network illustrated on the left hand side in Fig. 16. The contraction simplifies to the one depicted on the right hand side. For $\ell \equiv |x - x'| = |x' - x''| = |x'' - x| = 3^q$, we obtain the closed expression

$$\langle \hat{\sigma}_\alpha(x) \hat{\sigma}_\beta(x') \hat{\sigma}_\gamma(x'') \rangle_{\text{bound}} = \frac{C_{\alpha\beta\gamma}}{\ell^{\Delta_\alpha + \Delta_\beta + \Delta_\gamma}}, \quad (\text{E4})$$

where $C_{\alpha\beta\gamma} = \text{tr}[\hat{F}(\hat{\sigma}_\alpha \otimes \hat{\sigma}_\beta \otimes \hat{\sigma}_\gamma)]$.

Next, the corresponding bulk expectation value $\langle \hat{T}_{x,x',x''} \hat{\sigma}_\alpha(x, 0) \hat{\sigma}_\beta(x', 0) \hat{\sigma}_\gamma(x'', 0) \rangle_{\text{bulk}}$ is obtained by contracting the tensor network depicted in Fig. 17. We obtain the closed expression

$$\langle \hat{T}_{x,x',x''} \hat{\sigma}_\alpha(x, 0) \hat{\sigma}_\beta(x', 0) \hat{\sigma}_\gamma(x'', 0) \rangle_{\text{bulk}} = \frac{C'_{\alpha\beta\gamma}}{\ell^{\Delta_\alpha + \Delta_\beta + \Delta_\gamma}}, \quad (\text{E5})$$

where $C'_{\alpha\beta\gamma} = \text{tr}[\hat{F}'(\hat{\sigma}_\alpha \otimes \hat{\sigma}_\beta \otimes \hat{\sigma}_\gamma)]$. Comparing Eq. (E4) and Eq. (E5) we have

$$\langle \hat{\sigma}_\alpha(x) \hat{\sigma}_\beta(x') \hat{\sigma}_\gamma(x'') \rangle_{\text{bound}} = g(\hat{u}, \hat{w}, \hat{\sigma}_\alpha, \hat{\sigma}_\beta, \hat{\sigma}_\gamma) \times \langle \hat{T}_{x,x',x''} \hat{\sigma}_\alpha(x, 0) \hat{\sigma}_\beta(x', 0) \hat{\sigma}_\gamma(x'', 0) \rangle_{\text{bulk}}, \quad (\text{E6})$$

where $g(\hat{u}, \hat{w}, \hat{\sigma}_\alpha, \hat{\sigma}_\beta, \hat{\sigma}_\gamma) = C_{\alpha\beta\gamma}/C'_{\alpha\beta\gamma}$, which is Eq. (8).

3. Reyni entanglement entropy

Figure 18 illustrates a tensor network contraction that equates to the second Reyni entanglement entropy $S^{(2)}(|\Psi^{\text{bound}}\rangle, \mathcal{R}^{\text{bound}})$ of a block of boundary sites located at $x, x+1, \dots, x'$. In the limit of large $|x - x'|$ we obtain

$$S^{(2)}(|\Psi^{\text{bound}}\rangle, \mathcal{R}^{\text{bound}}) \approx \log \eta^{2 \log |x - x'|} + [\text{Tr}(\hat{\eta} \hat{E} \hat{\eta})]^2, \quad (\text{E7})$$

where η denotes the largest eigenvalue of transfer super-operator \hat{R} obtained by contracting two copies of \hat{w} and two copies of \hat{w}^* that are enclosed within a dashed contour in Fig. 18, $\hat{\eta}$ denotes the corresponding eigenoperator and \hat{E} is the same environment tensor that appears in Eq. (E1). Here η is related to the central charge of the CFT as $2 \log \eta = c/6$ [19]. Define the projected bulk state

$$|\Omega^{\text{bulk}}\rangle \equiv \hat{K}_{x,x'} |\Psi^{\text{bulk}}\rangle. \quad (\text{E8})$$

Denote by $\mathcal{R}^{\text{bulk}}$ the bulk region composed of sites enclosed between the geodesic extending between sites at $(x, 0)$ and $(x', 0)$ and the boundary, and including the bulk sites located at $(x+1, 0), (x+2, 0), \dots, (x'-1, 0)$. The second Reyni entanglement entropy $S^{(2)}(|\Omega^{\text{bulk}}\rangle, \mathcal{R}^{\text{bulk}})$ is obtained by contracting the tensor network illustrated in Fig. 19. In the limit of large $|x - x'|$, we obtain

$$S^{(2)}(\hat{K}_{x,x'} |\Psi^{\text{bulk}}\rangle, \mathcal{R}^{\text{bulk}}) \approx \log \eta^{2 \log |x - x'|} + [\text{Tr}(\hat{\eta} \hat{E}' \hat{\eta})]^2, \quad (\text{E9})$$

where \hat{E}' is the environment tensor that appears in Eq. (E2). Comparing Eq. (9) and Eq. (E9) we have

$$S^{(2)}(|\Psi^{\text{bound}}\rangle, \mathcal{R}^{\text{bound}}) = S^{(2)}(|\Omega^{\text{bulk}}\rangle, \mathcal{R}^{\text{bulk}}) + h(\hat{u}, \hat{w}), \quad (\text{E10})$$

where $h(\hat{u}, \hat{w}) = [\text{Tr}(\hat{\eta} \hat{E} \hat{\eta})]^2 - [\text{Tr}(\hat{\eta} \hat{E}' \hat{\eta})]^2$, which is Eq. (9). This derivation readily generalizes to higher Reyni entropies, $\alpha > 2$. We obtain

$$S^{(\alpha)}(|\Psi^{\text{bound}}\rangle, \mathcal{R}^{\text{bound}}) = S^{(\alpha)}(|\Omega^{\text{bulk}}\rangle, \mathcal{R}^{\text{bulk}}) + h^{(\alpha)}(\hat{u}, \hat{w}), \quad (\text{E11})$$

where $h^{(\alpha)}(\hat{u}, \hat{w}) = [\text{Tr}(\hat{\eta} \hat{E} \hat{\eta})]^\alpha - [\text{Tr}(\hat{\eta} \hat{E}' \hat{\eta})]^\alpha$.

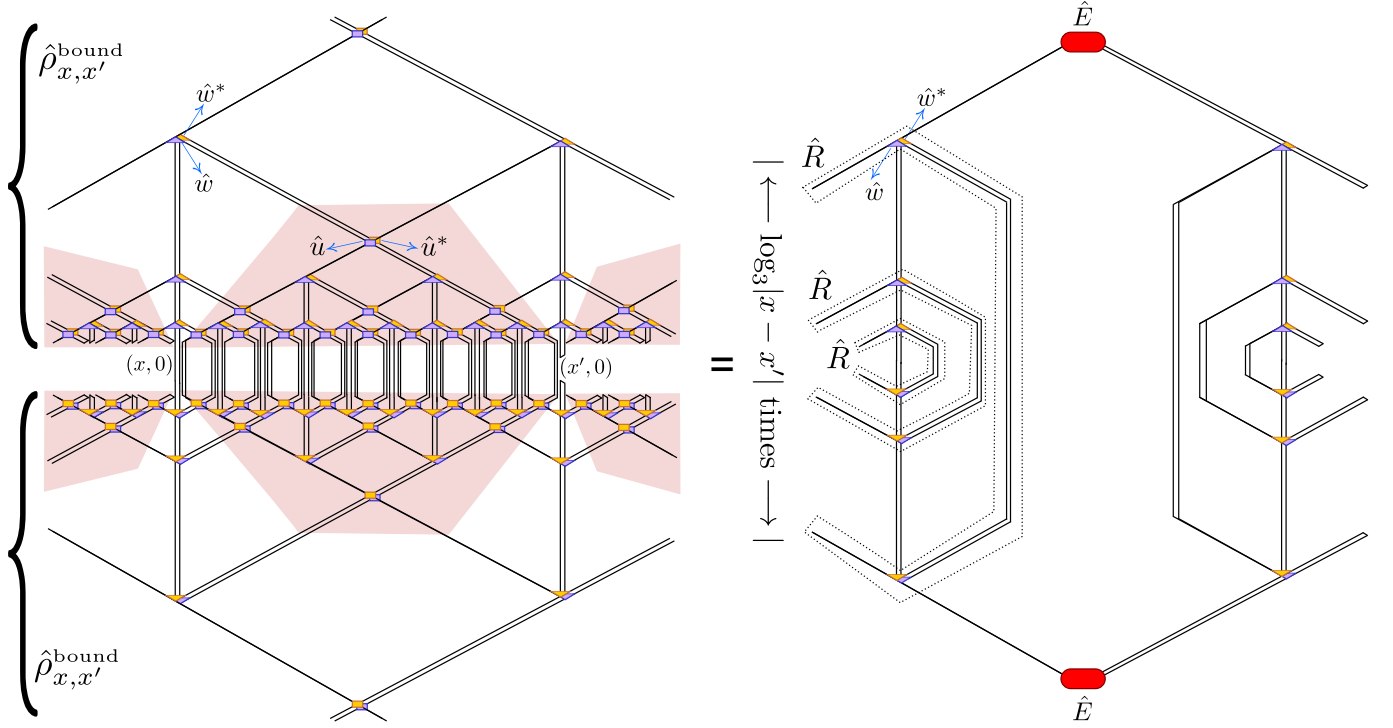


FIG. 18. (Color online) Tensor network contraction that equates to $[\text{Tr}(\hat{\rho}_{x,x'}^{\text{bound}})]^2$, where $\hat{\rho}_{x,x'}^{\text{bound}}$ is the boundary reduced density matrix of a block of $|x - x'|$ sites. The contraction consists of gluing together two copies of the tensor network representation of $\hat{\rho}_{x,x'}^{\text{bound}}$ along the open indices. Tensors located inside the red highlighted regions are pairwise multiplied with their Hermitian adjoints and cancel out. The resulting contraction (shown on the right) consists of $\log |x - x'|$ powers of a transfer superoperator \hat{R} (obtained by contracting two copies of \hat{w} and two copies of \hat{w}^* that are enclosed within a dashed contour) and the same environment tensor \hat{E} that appears in Fig. 14.

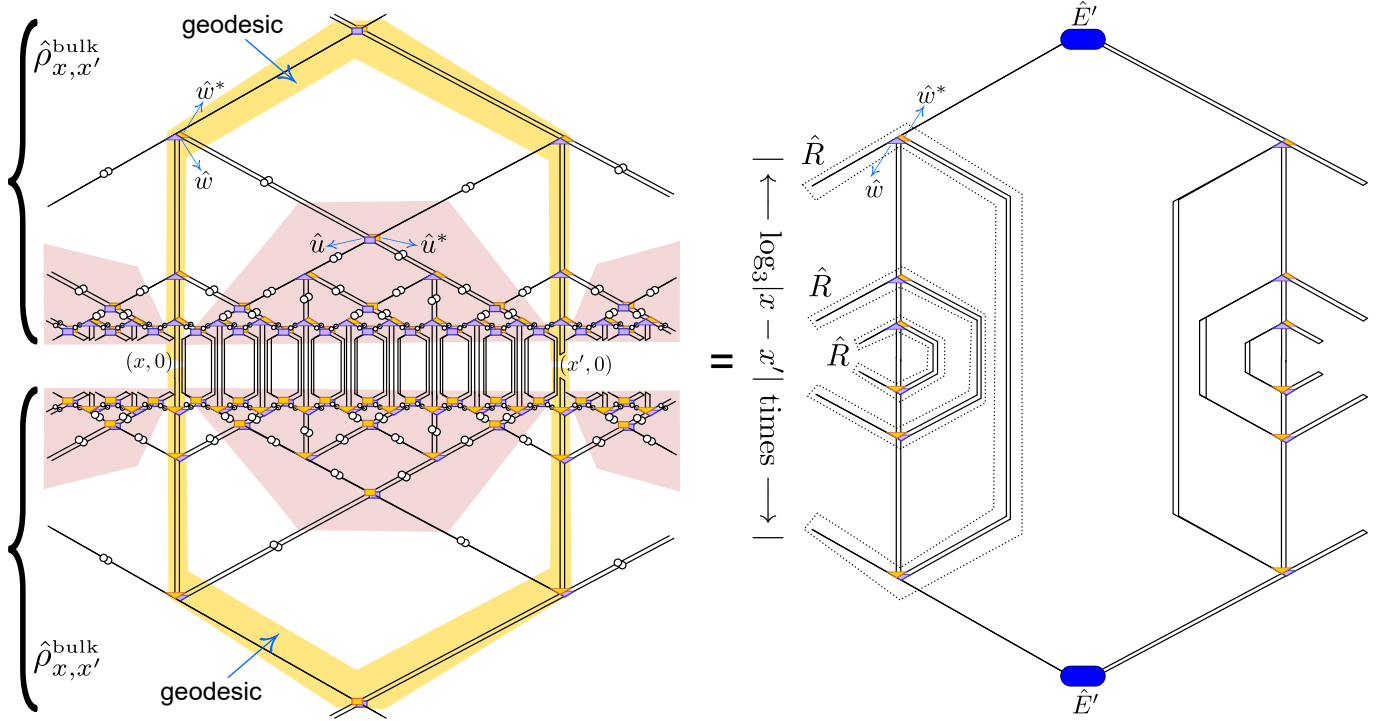


FIG. 19. (Color online) Tensor network contraction that equates to $[\text{Tr}(\hat{\rho}_{x,x'}^{\text{bulk}})]^2$, where $\hat{\rho}_{x,x'}^{\text{bulk}}$ is the bulk reduced density matrix of the bulk region $\mathcal{R}^{\text{bulk}}$ obtained from the tensor network representation of the projected bulk state $|\Omega^{\text{bulk}}\rangle$, Eq. (E8). The latter is obtained by applying the string projector $\hat{K}_{x,x'}$ on the bulk state $|\Psi^{\text{bulk}}\rangle$. Operator $\hat{K}_{x,x'}$ acts to remove the copy tensors located in the yellow highlighted regions. The contraction consists of gluing together two copies of the tensor network representation of $\hat{\rho}_{x,x'}^{\text{bulk}}$ along the open indices. Tensors located inside the red highlighted regions are pairwise multiplied with their Hermitian adjoints and cancel out. The resulting contraction (shown on the right) consists of $\log|x-x'|$ powers of the transfer superoperator \hat{R} that appears in Fig. 18 but a different environment tensor \hat{E}' (which also appears in Fig. 15).